Methods for Computing the Connectivity

1. The Issues in Minimizing F

In Georgoulis & Rust (2007), the connectivity is determined by minimizing the quantity

$$F = \sum_{i=1}^{m}\sum_{j=1}^{l}\left(\frac{|\boldsymbol{r}_i - \boldsymbol{r}_j|}{|\boldsymbol{r}_i| + |\boldsymbol{r}_j|} + \frac{|\Phi'_i + \Phi'_j|}{|\Phi'_i| + |\Phi'_j|}\right). \tag{1}$$

There are several problems with this approach:

- The connectivity which minimizes F is *not* necessarily the connectivity of the minimum energy (potential) magnetic field.
- The connectivity which minimizes F depends on the choice of origin for the coordinate system.

The second of these conditions by itself makes the resulting connectivity unphysical. In fact, the first condition automatically follows from the second, but since Georgoulis & Rust state "A minimum flux imbalance and overall separation length naturally suggest a minimum-energy solution for the studied magnetic structure." we provide simple examples to illustrate each of these issues.

1.1. Comparison with the Potential Field Connectivity

Consider a configuration of four flux-balanced sources, shown in Figure 1, consisting of two positive sources with flux Φ_0 located at $\boldsymbol{r}_{1,2} = (\pm 1, 0, 0)L$, and two negative sources with flux $-\Phi_0$ located at $\boldsymbol{r}_{3,4} = (\pm\sin\theta, \pm\cos\theta, 0)L$, with $0 \leq \theta < \pi/2$. For this configuration, $|\boldsymbol{r}_i| = L\,\forall i$, and

$$|\boldsymbol{r}_1 - \boldsymbol{r}_3| = \sqrt{L^2 + L^2 - 2L^2\cos\theta} = 2L\sin\theta/2 \tag{2}$$

$$|\boldsymbol{r}_2 - \boldsymbol{r}_4| = \sqrt{L^2 + L^2 - 2L^2\cos\theta} = 2L\sin\theta/2 \tag{3}$$

$$|\boldsymbol{r}_1 - \boldsymbol{r}_4| = \sqrt{L^2 + L^2 - 2L^2\cos(\pi - \theta)} = 2L\cos\theta/2 \tag{4}$$

$$|\boldsymbol{r}_2 - \boldsymbol{r}_3| = \sqrt{L^2 + L^2 - 2L^2\cos(\pi - \theta)} = 2L\cos\theta/2. \tag{5}$$

Since this configuration is exactly flux-balanced, we assume there are no open field lines (which in this method implies there are no connections between sources of the same polarity) and the second term in the expression for F then vanishes. If there are open field lines, then clearly the connectivity is not equal to the potential field connectivity in any case. Based on private communication with M. Georgoulis (2007), the method also requires that all the sources be connected by a single closed curve; the only closed curve connecting all the sources consists of connections 1–3, 3–2, 2–4 and 4–1 (not necessarily in that order), and the contribution to F from each of these connections is given by

$$\begin{aligned} F_{ij} &= \frac{|\boldsymbol{r}_i - \boldsymbol{r}_j|}{|\boldsymbol{r}_i| + |\boldsymbol{r}_j|} \\ F_{13} &= \sin\theta/2 \\ F_{32} &= \cos\theta/2 \\ F_{24} &= \sin\theta/2 \\ F_{41} &= \cos\theta/2 \end{aligned} \tag{6}$$

For the range of θ considered, one has $\sin\theta/2 \leq \cos\theta/2$, where the equality only holds for $\theta = 0$. Thus, except at $\theta = 0$, it is always more favorable for the connections 1–3 and 2–4 to be present; when $\theta = 0$, there is no unique connectivity that gives a minimum value of F since any division of the flux from source 1 between connections 1–3 and 1–4 will give the same value of F, and similarly for the other sources/connections.

Let $\Phi_{\rm min} = \Phi_0/n$, where n is a positive integer. In the method described by Georgoulis & Rust (2007), $n = 1$, since $\Phi_{\rm min}$ is chosen to equal the magnitude of the smallest source. In this case, we allow n to be greater than one, to demonstrate that this is not an artifact of choosing $\Phi_{\rm min} = |\Phi_i|\,\forall i$, but assume that the pixel size is much smaller than the distance between flux concentrations. In this case, the minimum value of F is obtained by including one connection each for 1–4 and 2–3, which is required for a closed curve, and having all the remaining connections between 1–3 and 2–4. Thus, the connectivity matrix is given by

$$\begin{aligned} \Phi_{13} &= (2n-1)\Phi_0/(2n) \\ \Phi_{32} &= \Phi_0/(2n) \\ \Phi_{24} &= (2n-1)\Phi_0/(2n) \\ \Phi_{41} &= \Phi_0/(2n). \end{aligned} \tag{7}$$

Note that this is independent of θ, while it does depend on n, although one could argue that the correct value is found by taking the limit $n \to \infty$, in which case $\Phi_{13} = \Phi_{24} = \Phi_0$ and $\Phi_{32} = \Phi_{41} = 0$. In comparison, the potential field connectivity, as determined by tracing field lines and using the Bayesian estimate for the flux given in Barnes, Longcope & Leka

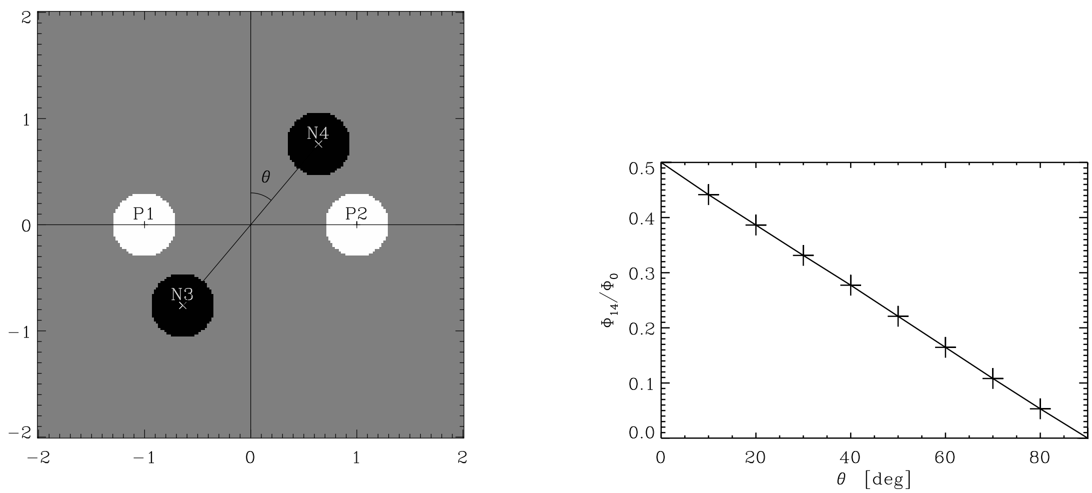

Fig. 1.— *Left:* The configuration of four sources uses to demonstrate that minimizing F does not recover the potential field connectivity. *Right:* The flux in one connection, Φ_{14}, as a function of source location, as determined by field line tracing, which varies linearly with the angle. In comparison, the connectivity from minimizing F maintains a fixed value of $\Phi_0/(2n)$ except at $\theta = 0$, where it is not uniquely determined.

(2005), varies linearly with θ as

$$\begin{aligned}
\Phi_{13} &= (1/2+\theta/\pi)\Phi_0 \\
\Phi_{32} &= (1/2-\theta/\pi)\Phi_0 \\
\Phi_{24} &= (1/2+\theta/\pi)\Phi_0 \\
\Phi_{41} &= (1/2-\theta/\pi)\Phi_0 \qquad (8)
\end{aligned}$$

and is shown in Figure 1. Thus, except at one particular value of θ (which depends on the value of n), the connectivity which minimizes F can *not* be the minimum energy connectivity.

1.2. Dependence on the Origin of the Coordinate System

Consider now a configuration of four flux concentrations, with no net flux, placed on the vertices of a square, with two positive concentrations with flux $3\Phi_0, \Phi_0$ centered at $\boldsymbol{r}_{1,2} = (x_0 \pm L, y_0, 0)$, and two negative concentrations each with flux $-2\Phi_0$ centered at $\boldsymbol{r}_{3,4} = (x_0, y_0 \pm L, 0)$. The relative positions of the sources does not change when shifting the origin, $(x_0, y_0, 0)$, thus there is no physical reason for the connectivity to change.

Represent the large positive concentration by three sources located at $\boldsymbol{r}_1 = (x_0 + L, y_0, 0), \boldsymbol{r}_{1\pm} = (x_0 + L \pm d, y_0, 0)$, and the two negative concentrations by two sources each, located at $\boldsymbol{r}_{3\pm} = (x_0 \pm d/2, y_0 + L, 0), \boldsymbol{r}_{4\pm} = (x_0 \pm d/2, y_0 - L, 0)$. In this case,

$$\begin{aligned}
|\boldsymbol{r}_1 - \boldsymbol{r}_{3\pm}| &= \sqrt{(L \mp d/2)^2 + L^2} \qquad (9)\\
|\boldsymbol{r}_1 - \boldsymbol{r}_{4\pm}| &= \sqrt{(L \mp d/2)^2 + L^2} \qquad (10)\\
|\boldsymbol{r}_{1\pm} - \boldsymbol{r}_{3\mp}| &= \sqrt{(L \pm 3d/2)^2 + L^2} \qquad (11)\\
|\boldsymbol{r}_{1\pm} - \boldsymbol{r}_{3\pm}| &= \sqrt{(L \pm d/2)^2 + L^2} \qquad (12)\\
|\boldsymbol{r}_{1\pm} - \boldsymbol{r}_{4\mp}| &= \sqrt{(L \pm 3d/2)^2 + L^2} \qquad (13)\\
|\boldsymbol{r}_{1\pm} - \boldsymbol{r}_{4\pm}| &= \sqrt{(L \pm d/2)^2 + L^2} \qquad (14)\\
|\boldsymbol{r}_2 - \boldsymbol{r}_{3\pm}| &= \sqrt{(L \pm d/2)^2 + L^2} \qquad (15)\\
|\boldsymbol{r}_2 - \boldsymbol{r}_{4\pm}| &= \sqrt{(L \pm d/2)^2 + L^2} \qquad (16)
\end{aligned}$$

and

$$\begin{aligned}
|\boldsymbol{r}_1| &= \sqrt{(x_0 + L)^2 + y_0^2} \qquad (17)\\
|\boldsymbol{r}_{1\pm}| &= \sqrt{(x_0 + L \pm d)^2 + y_0^2} \qquad (18)
\end{aligned}$$

$$
\begin{aligned}
|\boldsymbol{r}_2| &= \sqrt{(x_0-L)^2+y_0^2} &(19)\\
|\boldsymbol{r}_{3\pm}| &= \sqrt{(x_0\pm d/2)^2+(y_0+L)^2} &(20)\\
|\boldsymbol{r}_{4\pm}| &= \sqrt{(x_0\pm d/2)^2+(y_0-L)^2}. &(21)
\end{aligned}
$$

As in the previous case, we assume that only connections between sources of opposite polarity are present, thus we need only consider 1–3, 3–2, 2–4 and 4–1, although each of these connections can involve contributions from a different pair of sources. The contribution to F from each of these is given by

$$
\begin{aligned}
F_{13\pm} &= \frac{\sqrt{(L\mp d/2)^2+L^2}}{\sqrt{(x_0+L)^2+y_0^2}+\sqrt{(x_0\pm d/2)^2+(y_0+L)^2}}\\
F_{1\pm 3\pm} &= \frac{\sqrt{(L\pm d/2)^2+L^2}}{\sqrt{(x_0+L\pm d)^2+y_0^2}+\sqrt{(x_0\pm d/2)^2+(y_0+L)^2}}\\
F_{1\pm 3\mp} &= \frac{\sqrt{(L\pm 3d/2)^2+L^2}}{\sqrt{(x_0+L\pm d)^2+y_0^2}+\sqrt{(x_0\pm d/2)^2+(y_0+L)^2}}\\
F_{14\pm} &= \frac{\sqrt{(L\mp d/2)^2+L^2}}{\sqrt{(x_0+L)^2+y_0^2}+\sqrt{(x_0\pm d/2)^2+(y_0-L)^2}}\\
F_{1\pm 4\pm} &= \frac{\sqrt{(L\pm d/2)^2+L^2}}{\sqrt{(x_0+L\pm d)^2+y_0^2}+\sqrt{(x_0\pm d/2)^2+(y_0-L)^2}}\\
F_{1\pm 4\mp} &= \frac{\sqrt{(L\pm 3d/2)^2+L^2}}{\sqrt{(x_0+L\pm d)^2+y_0^2}+\sqrt{(x_0\pm d/2)^2+(y_0-L)^2}}\\
F_{23\pm} &= \frac{\sqrt{(L\pm d/2)^2+L^2}}{\sqrt{\sqrt{(x_0-L)^2+y_0^2}+(x_0\pm d/2)^2+(y_0+L)^2}}\\
F_{24\pm} &= \frac{\sqrt{(L\pm d/2)^2+L^2}}{\sqrt{\sqrt{(x_0-L)^2+y_0^2}+(x_0\pm d/2)^2+(y_0-L)^2}} \qquad (22)
\end{aligned}
$$

With $\Phi_{\rm min}=\Phi_0$, subject to the constraint of a closed curve, there are only three distinct connectivities possible for this configuration: $\Phi_{13}=\Phi_{14}=3\Phi_0/2$, $\Phi_{23}=\Phi_{24}=\Phi_0/2$ (Case 1, which is the potential field connectivity), $\Phi_{13}=\Phi_0$, $\Phi_{14}=2\Phi_0$, $\Phi_{23}=\Phi_0$, $\Phi_{24}=0$ (Case 2) and $\Phi_{13}=2\Phi_0$, $\Phi_{14}=\Phi_0$, $\Phi_{23}=0$, $\Phi_{24}=\Phi_0$ (Case 3). Each of these configurations can be obtained with multiple different ways of connecting the sources.

In Figure 2, we show as a function of x_0, y_0 and for two values of d/L which of the three cases has a minimum value for F. In the limit of zero pixel size, the connectivity which minimizes F changes whenever either x_0 or y_0 changes sign such that when $x_0y_0>0$, Case 2 minimizes F, while when $x_0y_0<0$, Case 3 minimizes F; when either $x_0=0$ or $y_0=0$, then there is no unique connectivity which minimizes F. For finite pixel sizes, there is an area of parameter space about each of the axes for which Case 1 minimizes F. In this case, the connectivity which minimizes F is a function of *both* the origin of the coordinate system and the size of a pixel.

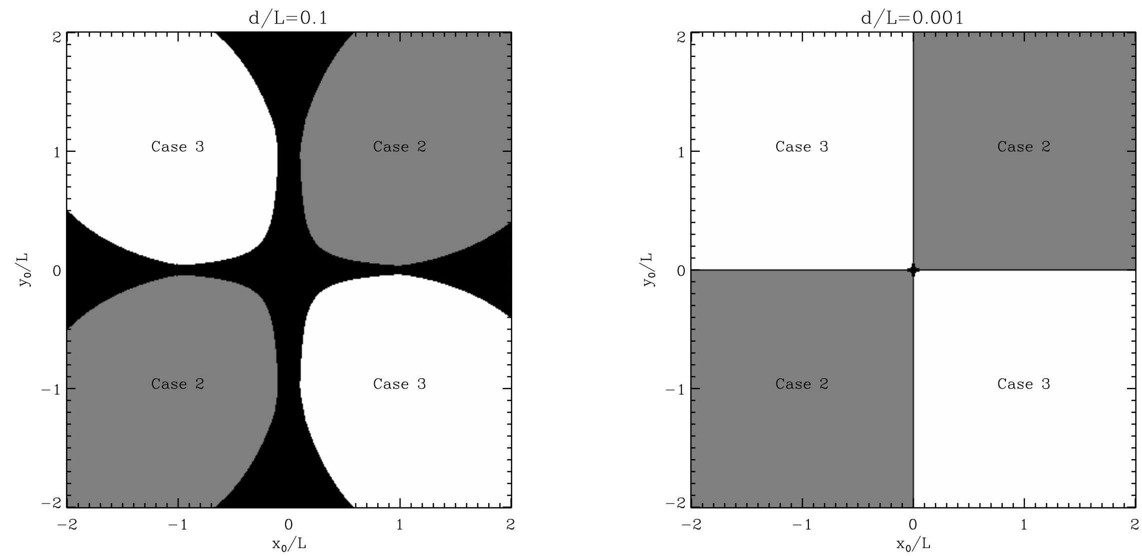

Fig. 2.— The connectivity which minimizes F as a function of the location of the origin of the coordinate system in which it is calculated for a large pixel size $d/L = 0.1$ (left) and a small pixel size $d/L = 0.001$ (right). Not only does the connectivity depend on the origin of the coordinate system, but it also depends on the size of a pixel relative to the spacing between flux concentrations.