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# On the symmetrization rate of an intense geophysical vortex $\stackrel{\text{tr}}{\sim}$

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# Abstract

Numerical models demonstrate that a broad class of geophysical vortices freely evolve toward vertically aligned, axisymmetric states. In principle, this intrinsic drive toward symmetry opposes destructive shearing by the environmental flow.

This article examines the case in which a discrete vortex-Rossby-wave dominates a perturbation from symmetry, and symmetrization occurs by decay of the wave. The wave is damped by a resonance with the fluid rotation frequency at a critical radius,  $r_*$ . The damping rate is proportional to the radial derivative of potential vorticity at  $r_*$ . Until now, the theory of resonantly damped vortex-Rossby-waves (technically quasi-modes) was formally restricted to slowly rotating vortices, which obey quasigeostrophic (QG) dynamics. This article extends the theory to rapidly rotating vortices.

The analysis makes use of the asymmetric balance (AB) approximation. Even at a modest Rossby number (unity), AB theory can predict damping rates that exceed extrapolated QG results by orders of magnitude. This finding is verified upon comparison of AB theory to numerical experiments, based on the primitive equations. The experiments focus on the decay of low azimuthal wave-number asymmetries.

A discrete vortex-Rossby-wave can also resonate with an outward propagating inertia-buoyancy wave (Lighthill radiation), inducing both to *grow*. At large Rossby numbers, this growth mechanism can be dynamically relevant. All balance models, including AB theory, neglect inertia-buoyancy waves, and therefore ignore the possibility of a Rossby-inertia-buoyancy (RIB) instability. This article shows that a large potential vorticity gradient (of the proper sign) at the critical radius  $r_*$  can suppress the RIB instability, and thereby preserve balanced flow, even at large Rossby numbers. © 2003 Elsevier Science B.V. All rights reserved.

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Fig. 1. In time, a geophysical vortex can become (a) vertically aligned, and (b) axisymmetric by intrinsic means.

#### 1. Introduction

Environmental shear-flow persistently disturbs geophysical vortices, such as tropical cyclones and ocean eddies. Such shear will tilt, distort and possibly destroy a vortex. However, a vortex may limit the impact of external shear by its intrinsic drive toward a vertically aligned, horizontally axisymmetric state.

Fig. 1 illustrates both vertical alignment and horizontal axisymmetrization. During alignment, the centers of rotation at all heights spiral into a central vertical axis. During axisymmetrization, the flow in a horizontal plane becomes circular. Detailed descriptions of alignment are presented, for example, in Polvani (1991), Viera (1995), and Sutyrin et al. (1998). Similar descriptions of two-dimensional axisymmetrization may be found in Melander et al. (1987), and Dritschel (1998). Reasor and Montgomery (2001), Enagonio and Montgomery (2001), Moller and Montgomery (1999) and Montgomery and Enagonio (1998) discuss how alignment and axisymmetrization help maintain and intensify a tropical cyclone.

One important, conservative mechanism for symmetrization<sup>1</sup> is a resonant wave-fluid interaction. In this case, the perturbation from symmetry is dominated by one or more discrete vortex-Rossby-waves. Each wave is damped by a resonance with the bulk fluid rotation frequency at a critical radius,  $r_*$ . The damping rate is proportional to the radial derivative of the azimuthally averaged potential vorticity at  $r_*$ . Briggs et al. (1970; hereafter BDL) first developed the theory of resonantly damped vortex-Rossby-waves in order to predict the rate at which an ideal two-dimensional vortex becomes axisymmetric. Recent laboratory experiments have tested and verified this rate (Schecter et al., 2000;

<sup>&</sup>lt;sup>1</sup> In this article, the term "symmetrization" refers to both vertical alignment and horizontal axisymmetrization.

Pillai and Gould, 1994). Schecter et al. (2002; hereafter SMR) extended the work of BDL to three-dimensional quasigeostrophic (QG) flow. In doing so, they confirmed that resonant damping can also drive vertical alignment.

Until now, the theory of resonantly damped vortex-Rossby-waves was formally restricted to cases in which  $R_0 \ll 1$ . Here,  $R_0$  is the ratio of relative to planetary vorticity, i.e. the Rossby number. The small Rossby number domain does not apply to hurricanes, nor even tropical depressions. The present article develops a theory of resonantly damped vortex-Rossby-waves that extends to  $R_0 \gtrsim 1$ . We will restrict our attention to waves in a barotropic vortex on the *f*-plane, in a uniformly stratified atmosphere/ocean. Because the unperturbed vortex is barotropic, its azimuthal velocity profile,  $\bar{v}(r)$ , has no vertical dependence. This simplified basic-state is adequate, in our opinion, for illustrating the damping rate versus  $R_0$ .

Our analysis will assume that the vortex maintains hydrostatic balance, and furthermore that the vortex maintains asymmetric balance (Shapiro and Montgomery, 1993). Hydrostatic balance formally requires that

$$\mathcal{D}_N^2 \equiv \frac{D_V^2/Dt^2}{N^2} \ll 1,\tag{1}$$

and asymmetric balance further requires that

$$\mathcal{D}_I^2 \equiv \frac{D_V^2 / Dt^2}{\bar{\eta}\bar{\xi}} \ll 1.$$
<sup>(2)</sup>

Here,  $D_V/Dt = \partial/\partial t + \bar{v}r^{-1}\partial/\partial\lambda$ , where *t* is time and  $\lambda$  is the polar angle (see Fig. 1). In addition, *N* is the local buoyancy frequency,  $\bar{\eta}(r)$  is the local absolute vorticity of the mean flow, and  $\bar{\xi}(r)$  is the average absolute vorticity within the radius *r*. We emphasize that  $\mathcal{D}_I^2$  can be much less than unity, even if  $R_0 > 1$ . This is because  $(D_V/Dt)^{-1}$  is the time scale for change in a *moving* reference frame, specifically, one that moves with the local mean flow. Furthermore,  $N^2 > \bar{\eta}\bar{\xi}$  for many atmospheric and oceanic applications; hence, (2) typically implies (1). Tornadoes provide a notable exception to this rule.

Unlike traditional balance (e.g. McWilliams, 1985), asymmetric balance (AB) permits an order unity ratio of perturbation divergence to perturbation vorticity. As a result, the AB approximation may better characterize asymmetric disturbances that are observed in tropical cyclones (Montgomery and Franklin, 1998).

Using a primitive equation model, we will examine the alignment and/or axisymmetrization of several vortices: a Rankine-with-skirt vortex, a Gaussian vortex, and a convection-free approximation of Hurricane Olivia (1994). At the beginning of each simulation, the vortex is either tilted, or distorted elliptically. In all cases considered, the perturbation is quickly dominated by a single discrete vortex-Rossby-wave. For cases in which  $D_I^2 \ll 1$  at all r, the wave decays at the rate predicted by the AB theory of resonant damping. It will be shown that this rate can be orders of magnitude greater than that predicted by a naive extrapolation of QG theory to large Rossby numbers. Such accelerated symmetrization was hypothesized earlier by Reasor and Montgomery (2001).

At very large Rossby numbers, and large azimuthal wave-numbers, it is difficult to satisfy  $\mathcal{D}_I^2 \ll 1$  at all r. If this condition is not uniformly satisfied, AB theory may provide misleading results. This will be illustrated by a particular case, in which AB theory predicts a weakly damped (or neutral) vortex-Rossby-wave, but the primitive equation model yields an exponentially growing wave. The growth is caused by a resonant coupling to an outward-propagating, spiral, inertia-buoyancy wave (Ford, 1994). Such inertia-buoyancy waves are filtered out of the AB model, and balance models in general. We will show that the resonant wave–fluid interaction inhibits the instability (sometimes called Lighthill radiation), if the mean flow potential vorticity gradient is sufficiently large (and of the proper sign) at the critical radius  $r_*$ . The consequent damping agrees quantitatively with AB theory if  $\mathcal{D}_I^2 \ll 1$  at least in the vortex core.

Before starting the main text, we note the following. To begin with, there are cases in which the resonant wave–fluid interaction causes a discrete vortex-Rossby-wave to grow, as opposed to decay. Decay occurs only if the radial derivative of potential vorticity has the same sign at  $r_*$  as it does in the vortex core. If the potential vorticity decreases or increases monotonically with radius, the damping condition is automatically satisfied. For this reason, we will focus on monotonic vortices.

Furthermore, resonantly damped vortex-Rossby-waves are not normal modes of the perturbation equations, rather, they are quasi-modes. As a result, we cannot use a standard eigenmode analysis to compute their complex frequencies. Instead, we will use methods that are outlined in BDL.

Finally, quasi-modes do not form a complete set of solutions to the initial value problem. Accordingly, there are cases in which quasi-modes contribute little to the perturbation, and do not control symmetrization. Instead, the vortex may become vertically aligned, and horizontally axisymmetric, by the wind-up and radiation of "sheared vortex-Rossby-waves" (e.g. Montgomery and Kallenbach, 1997; Bassom and Gilbert, 1998; Reasor and Montgomery, 2001; Brunet and Montgomery, 2002). The present article does not fully resolve the conditions that favor one form of symmetrization over the other. This issue will be addressed in forthcoming papers.

The remainder of this article is organized as follows. Section 2 describes the perturbation equations and the AB approximation. Section 3 derives an expression for the damping rate of a discrete vortex-Rossby-wave, using AB theory. Section 4 compares this expression to the symmetrization rate of a vortex in a primitive equation model. Section 4 also includes a discussion of the Rossby-inertia-buoyancy instability. Section 5 recapitulates the main results, and discusses their possible relevance to the robustness of tropical cyclones.

#### 2. Basic state and perturbation equations

In this section we will describe the class of basic vortex states under consideration, and the equations that govern perturbations about those states. Our discussion here, and hereafter, will focus on atmospheric vortices. However, our results for the symmetrization rate versus Rossby number and internal deformation radius apply equally well to oceanic vortices, which obey isomorphic dynamics (e.g. Pedlosky, 1987).

To begin with, we introduce a cylindrical coordinate system  $(r, \lambda, z)$ , in which z is a pressure-based pseudo-height (Hoskins and Bretherton, 1972). Increments of pseudo-height are related to increments of physical height  $z_*$  by  $\theta_{\text{tot}} dz = \theta_0 dz_*$ . Here,  $\theta_{\text{tot}}$  and  $\theta_0$  are the

variable and reference potential temperatures, respectively. For an adiabatic atmosphere,  $\theta_{\text{tot}} = \theta_0$ , and  $z = z_*$ . The subscript 'tot' is used here to denote a total field, which is the sum of basic state (overscore) and perturbation (undressed) fields; e.g.  $\theta_{\text{tot}} = \bar{\theta} + \theta$ .

The basic state of the vortex is barotropic, and characterized by the azimuthal velocity  $\bar{v}(r)$ . The following variables, related to  $\bar{v}$ , are used throughout this article for notational convenience:

$$\bar{\Omega}(r) = \frac{\bar{v}}{r}, \qquad \bar{\zeta}(r) = \frac{1}{r} \frac{\partial(r\bar{v})}{\partial r},$$

$$\bar{\xi}(r) = f + 2\bar{\Omega}, \quad \bar{\eta}(r) = f + \bar{\zeta}.$$
(3)

Here, f is the local Coriolis parameter,  $\overline{\Omega}$  is the unperturbed rotation frequency, and  $\overline{\zeta}$  is the unperturbed relative vorticity. The average and absolute vorticities,  $\overline{\xi}$  and  $\overline{\eta}$ , were introduced in Section 1. The Rossby number  $R_0$  of the basic state is defined here in terms of the relative vorticity and Coriolis parameter as follows:

$$R_{\rm o} = \frac{\zeta(0)}{f}.\tag{4}$$

The unperturbed geopotential  $\overline{\phi}(r, z)$  is in gradient balance with the azimuthal wind; i.e.

$$\frac{\partial \phi}{\partial r} = f \bar{v} + \frac{\bar{v}^2}{r}.$$
(5)

As the Rossby number approaches zero, the Coriolis term  $(f\bar{v})$  dominates the right-hand-side, and Eq. (5) reduces to geostrophic balance. As the Rossby number approaches infinity, the centrifugal term  $(\bar{v}^2/r)$  dominates the right-hand-side, and Eq. (5) reduces to cyclostrophic balance.

We will assume that the vortex maintains hydrostatic balance at all times. In the pseudoheight coordinate system, hydrostatic balance implies that

$$\frac{\partial \phi}{\partial z} = \frac{g}{\theta_0} \bar{\theta}, \qquad \frac{\partial \phi}{\partial z} = \frac{g}{\theta_0} \theta.$$
 (6)

Here,  $\phi$  is the perturbation geopotential, and g is the gravitational acceleration.

With hydrostatic balance, perturbations to the basic state are governed by the following primitive equations:

$$\frac{D_V u}{Dt} - \bar{\xi}v = -\frac{\partial\phi}{\partial r},\tag{7}$$

$$\frac{D_V v}{Dt} + \bar{\eta} u = -\frac{1}{r} \frac{\partial \phi}{\partial \lambda},\tag{8}$$

$$w = -\frac{1}{N^2} \frac{D_V}{Dt} \frac{\partial \phi}{\partial z},\tag{9}$$

$$\frac{1}{r}\frac{\partial(ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \lambda} + \frac{\partial w}{\partial z} = 0.$$
(10)

Here, (7) and (8) are the horizontal momentum equations, (9) is the adiabatic heat equation, and (10) is the mass continuity equation in Boussinesq form. The variables u, v and w

denote the radial, azimuthal and vertical velocity perturbations, respectively. The vertical velocity component is defined as the material derivative of the pseudoheight *z*. The buoyancy frequency N(z) is given by  $\sqrt{\partial^2 \bar{\phi} / \partial z^2}$ . To simplify the following analysis, *N* will be treated as a constant. This approximation is valid insofar as fractional changes of *N* are small between the top and bottom of the vortex. Note that Eqs. (7)–(10) neglect quadratic terms in the perturbation fields. In Section 5, we will briefly address the extent to which such linearization is valid.

It will prove useful to consider the evolution of potential vorticity. The basic state and perturbation potential vorticities are given by

$$\bar{\Pi} = N^2 \bar{\eta}, \qquad \Pi = N^2 \left[ \frac{1}{r} \frac{\partial(rv)}{\partial r} - \frac{1}{r} \frac{\partial u}{\partial \lambda} \right] + \bar{\eta} \frac{\partial^2 \phi}{\partial z^2}.$$
(11)

Here, it is worth noting that the radial derivatives of  $\overline{\Pi}$ ,  $\overline{\eta}$  and  $\overline{\zeta}$  have the same sign, and are directly proportional. Hence, the potential, absolute and relative vorticity gradients of the basic state are essentially synonymous. The tendency equation for  $\Pi$  is

$$\frac{D_V \Pi}{Dt} + N^2 u \frac{d\bar{\eta}}{dr} = 0.$$
<sup>(12)</sup>

Eq. (12) is derived by cross-differentiation of (7) and (8), and elimination of the divergence term using (9) and (10).

As discussed in the introduction, if  $\mathcal{D}_I^2$  is much less than unity, we may use an AB model to approximate the primitive equations (7)–(10). AB models can extend to arbitrary order in the small parameter  $\mathcal{D}_I^2$  (Shapiro and Montgomery, 1993). For simplicity, we will use a zero-order model. In this case, horizontal velocities are diagnostic variables, given by

$$u = -\frac{1}{\bar{\eta}} \left[ \frac{1}{r} \frac{\partial \phi}{\partial \lambda} + \frac{1}{\bar{\xi}} \frac{D_V}{Dt} \frac{\partial \phi}{\partial r} \right],\tag{13}$$

$$v = \frac{1}{\bar{\xi}} \left[ \frac{\partial \phi}{\partial r} - \frac{1}{\bar{\eta}r} \frac{D_V}{Dt} \frac{\partial \phi}{\partial \lambda} \right]. \tag{14}$$

Eqs. (13) and (14) are obtained by neglecting  $\mathcal{D}_I^2 u$  and  $\mathcal{D}_I^2 v$  in  $(\bar{\eta}\bar{\xi})^{-1}D_V/Dt \times$  (7) and (8), respectively.

Two paths have been followed toward a prognostic equation for the geopotential perturbation  $\phi$ . Shapiro and Montgomery (1993) substituted (9), (13) and (14) into the mass continuity equation (10). McWilliams et al. (2003) substituted (13) and (14) into the potential vorticity equation (12). Both paths yield dynamics of the form

$$\frac{D_V}{Dt}q + \frac{1}{r}\frac{\mathrm{d}\bar{\eta}^{-1}}{\mathrm{d}r}\frac{\partial\phi}{\partial\lambda} = 0,\tag{15}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\frac{r}{\bar{\eta}\bar{\xi}}\frac{\partial}{\partial r} + \frac{S(r)}{\bar{\eta}\bar{\xi}r^2}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{N^2}\frac{\partial^2}{\partial z^2}\right)\phi = q,\tag{16}$$

in which  $q(r, \lambda, z, t)$  is a pseudo potential vorticity perturbation, introduced for analytical convenience. The continuity and potential vorticity (PV) paths are distinguished

by *S*(*r*):

$$S(r) \equiv \begin{cases} 1 & \text{continuity path,} \\ \frac{(2\bar{\xi} - \bar{\eta})}{\bar{\eta}} & \text{PV path.} \end{cases}$$
(17)

If  $\overline{\zeta}(r)/f$  and  $\overline{\Omega}(r)/f$  are negligible, S(r) = 1 in both cases. In the same limit, q and  $\phi$  are the quasigeostrophic potential vorticity and streamfunction perturbations, respectively. More generally, S(r) = 1 in both cases if the differential rotation is negligible, in the sense that  $r\overline{\xi}^{-1}d\overline{\Omega}/dr \ll 1$ . We have compared both prognostic equations to numerical solutions of the primitive equations for several test cases in which  $\mathcal{D}_I^2 \ll 1$ . The prognostic equation obtained from the potential vorticity path gave better accuracy. Accordingly, the AB results presented here were obtained by using  $S(r) = (2\overline{\xi} - \overline{\eta})/\overline{\eta}$ .

Eqs. (15) and (16) form a closed set if spatial boundary conditions are imposed on  $\phi$ . In rough agreement with tropical cyclone observations (e.g. Hawkins and Rubsam, 1968), we will assume isothermal ( $\theta = 0$ ) boundary conditions at the top (z = H) and bottom (z = 0) surfaces of the vortex. From the hydrostatic relation, and the adiabatic heat equation, this implies that  $\partial \phi / \partial z$  and w both vanish at z = 0 and H. We will also assume that  $\phi$  is regular at the origin, and vanishes as r approaches infinity.

With such boundary conditions, Eqs. (15) and (16) conserve the total wave activity (Ren, 1999), defined by

$$\mathcal{A} = \int_{0}^{H} dz \int_{-\pi}^{\pi} d\lambda \int_{0}^{\infty} dr \frac{r^{2}}{d\bar{\eta}^{-1}/dr} q^{2}.$$
 (18)

Conservation of A provides a powerful constraint on the dynamics, which we will use in Section 3 to explain symmetrization.

Before delving into the symmetrization of a vortex, it will prove useful to expand q and  $\phi$  into the following Fourier series:

$$q = \sum_{m=0,n=1}^{\infty} q^{(m,n)}(r,t) \cos\left(\frac{m\pi z}{H}\right) e^{in\lambda} + \text{c.c.},$$
(19)

$$\phi = \sum_{m=0,n=1}^{\infty} \phi^{(m,n)}(r,t) \cos\left(\frac{m\pi z}{H}\right) e^{in\lambda} + \text{c.c.},$$
(20)

where c.c. denotes the complex conjugate. The n = 1 (m > 0) components of the expansion constitute misalignments (Fig. 1a), whereas the  $n \ge 2$  components constitute horizontal deformations (Fig. 1b).

In linearized dynamics, each Fourier component evolves independently. From Eq. (15), we have

$$\left(\frac{\partial}{\partial t} + in\bar{\Omega}\right)q^{(m,n)} = -i\frac{n}{r}\frac{d\bar{\eta}^{-1}}{dr}\phi^{(m,n)},\tag{21}$$

and from Eq. (16),

 $\sim$ 

$$\phi^{(m,n)} = f^2 \int_0^\infty \mathrm{d}r' r' G_{mn}(r,r') q^{(m,n)}(r',t).$$
(22)

Here,  $G_{mn}$  is a dimensionless Green function defined by

$$f^{2}\left(\frac{1}{r}\frac{\partial}{\partial r}\frac{r}{\bar{\eta}\bar{\xi}}\frac{\partial}{\partial r} - \frac{S(r)n^{2}}{\bar{\eta}\bar{\xi}r^{2}} - \frac{1}{f^{2}l_{\mathrm{R}}^{2}}\right)G_{mn} = \frac{\delta(r-r')}{r},$$
(23)

with the boundary conditions  $G_{mn}(0, r') = G_{mn}(\infty, r') = 0$ . In (23),  $\delta$  is a Dirac distribution, and we have introduced the (ambient) internal Rossby deformation radius,

$$l_{\rm R} = NH/m\pi |f|. \tag{24}$$

Note that  $G_{mn}$  depends on the velocity profile  $\bar{v}(r)$ ; i.e. it does not have a universal structure. Appendix A outlines the numerical construction of  $G_{mn}$  for an arbitrary profile.

# 3. Linear AB theory of discrete vortex-Rossby-waves and the resonant wave-fluid interaction

The AB model filters out "high-frequency" inertia-buoyancy waves. However, it retains vortex-Rossby-waves, i.e. waves supported by a gradient of basic-state potential vorticity. Below, we present the AB theory of discrete vortex-Rossby-waves.

As in SMR, we begin by considering a Rankine vortex:

$$\bar{\zeta}(r) = \begin{cases} Z_{0} & r < r_{v} \\ 0 & r > r_{v} \end{cases}, \qquad \bar{\Omega}(r) = \begin{cases} \frac{Z_{0}}{2} & r < r_{v} \\ \frac{Z_{0}r_{v}^{2}}{2r^{2}} & r > r_{v} \end{cases},$$
(25)

in which  $Z_0$  is a constant frequency. Without loss of generality, we will assume that  $Z_0 > 0$ . Such vortices are cyclones in the northern hemisphere, and anticyclones in the southern hemisphere. The Rankine vortex supports a set of discrete vortex-Rossby-waves of the form

$$q = aQ(r)\cos\left(\frac{m\pi z}{H}\right)e^{i(n\lambda-\omega t)} + \text{c.c.},$$
(26)

$$\phi = a\Phi(r)\cos\left(\frac{m\pi z}{H}\right)e^{i(n\lambda-\omega t)} + \text{c.c.},$$
(27)

where a is a dimensionless wave amplitude. One may verify, using (21), (22) and (25), that the radial eigenfunctions are given by

$$Q(r) = r_v \delta(r - r_v), \qquad \Phi(r) = f^2 r_v^2 G_{mn}^{(R)}(r, r_v), \tag{28}$$

up to a common multiplicative constant. Furthermore, the eigenfrequency is given by

$$\omega = Z_{\rm o} \left[ \frac{n}{2} + \frac{n}{1 + R_{\rm o}} G_{mn}^{(\rm R)}(r_{\nu}, r_{\nu}) \right],\tag{29}$$

where  $R_0 = Z_0/f > -1$ . In (28) and (29),  $G_{mn}^{(R)}$  denotes the Green function of a Rankine vortex. This Green function is negative and depends implicitly on  $l_R$  and  $R_0$ : Appendix A provides further details.



Fig. 2. The vorticity profile of a generic monotonic geophysical vortex. The vortex has two regions: a core, containing the bulk of the vorticity, and a skirt (shaded). The core alone can support discrete vortex-Rossby-waves. The critical radii  $(r_*)$  of these waves are presumed to be in the skirt.

The critical radius  $r_*$  of a wave is where the angular velocity of the mean vortex equals the phase-velocity of that wave. That is,  $r_*$  satisfies the resonance condition

$$\bar{\Omega}(r_*) = \frac{\omega}{n}.\tag{30}$$

For a Rankine vortex,  $r_* > r_v$ . This follows from the fact that  $G_{mn}^{(R)}(r_v, r_v) < 0$ , making the phase-velocity of the wave,  $\omega/n$ , less than the core rotation frequency,  $Z_0/2$ .

Of course, a Rankine vortex is a highly idealized representation of a geophysical vortex. A more realistic model will have a continuous vorticity distribution, as illustrated by Fig. 2. Such vortices can also support discrete vortex-Rossby-waves. By analogy to the Rankine vortex, we assume that the pseudo potential vorticity perturbation, q, associated with such a wave is concentrated in the core of the vortex, where the magnitude of  $d\bar{\zeta}/dr$  is maximal. So, if the outer skirt of low relative vorticity were removed, the discrete vortex-Rossby-waves would still exist.

By further analogy to a Rankine vortex, we assume that the critical radius  $r_*$  of a discrete vortex-Rossby-wave is finite, and outside the core. The skirt generally provides a vorticity gradient at  $r_*$ , which did not exist for the case of a Rankine vortex. Because of the resonance at  $r_*$ , even a slight vorticity gradient there profoundly affects the evolution of the wave amplitude.

To analyze the resonant interaction, it is convenient to decompose the pseudo potential vorticity perturbation into core (c) and skirt (s) contributions:  $q = q_c + q_s$ . We may also divide the total wave-activity into regional components:  $\mathcal{A} = \mathcal{A}_c + \mathcal{A}_s$ . These components are defined by

$$\mathcal{A}_{c} = \int_{0}^{H} dz \int_{-\pi}^{\pi} d\lambda \int_{0}^{r_{v}} dr \frac{r^{2}}{d\bar{\eta}^{-1}/dr} q_{c}^{2},$$
  
$$\mathcal{A}_{s} = \int_{0}^{H} dz \int_{-\pi}^{\pi} d\lambda \int_{r_{v}}^{\infty} dr \frac{r^{2}}{d\bar{\eta}^{-1}/dr} q_{s}^{2},$$
(31)

in which  $r_v$  is the radius of the vortex core.<sup>2</sup> By conservation of total wave activity, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_{\mathrm{c}} = -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_{\mathrm{s}}.\tag{32}$$

<sup>&</sup>lt;sup>2</sup> For the case of a Rankine vortex, one should interpret the integration limit  $r_v$  as just beyond the radius where  $\bar{\zeta}$  is discontinuous.

Suppose that at t = 0 a pseudo potential vorticity perturbation is created exclusively in the core region of the vortex. Suppose also that  $q_c$  is composed entirely of a single discrete vortex-Rossby-wave. The geopotential (pressure) perturbation of this core wave extends into the skirt. In time, the geopotential perturbation will cause  $q_s^2$  to increase, most effectively at  $r_*$ . Consider a case in which  $d\bar{\eta}^{-1}/dr|_{r_*}$  is positive. Then,  $\mathcal{A}_s$  increases with  $q_s^2$  at  $r_*$ . Further assume that  $d\bar{\eta}^{-1}/dr$  is positive throughout the core. Then, to conserve total  $\mathcal{A}, q_c^2$  must decrease; i.e. the core wave must decay. By similar reasoning, if  $d\bar{\eta}^{-1}/dr|_{r_*}$ is negative,  $\mathcal{A}_s$  will decrease, and the core wave will grow.

Conservation of wave activity (32) is now used to obtain a quantitative prediction of the decay rate of a discrete vortex-Rossby-wave. Suppose, once again, that  $q_c$  is composed exclusively of a discrete vortex-Rossby-wave, of the form given by (26), with  $\omega$  real, and *a* now a function of time. Then, by straightforward calculation, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_{\mathrm{c}} = 2\pi H \langle Q, Q \rangle \frac{\mathrm{d}|a|^2}{\mathrm{d}t},\tag{33}$$

where

$$\langle Q, Q \rangle = \int_0^{r_v} \mathrm{d}r \frac{r^2}{\mathrm{d}\bar{\eta}^{-1}/\mathrm{d}r} Q(r) Q^*(r),$$
 (34)

and the superscript '\*' denotes the complex conjugate. We will refer to  $\langle Q, Q \rangle$  as the norm of the wave. If  $d\bar{\eta}^{-1}/dr$  is positive in the vortex core, then the norm is positive.

The rate of change of wave activity in the skirt can be written as follows,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_{\mathrm{s}} = 4\pi H \int_{r_{v}}^{\infty} \mathrm{d}r \frac{r^{2}}{\mathrm{d}\bar{\eta}^{-1}/\mathrm{d}r} Re\left[q_{\mathrm{s}}^{(m,n)}\frac{\partial}{\partial t}q_{\mathrm{s}}^{(m,n)*}\right],\tag{35}$$

where Re[...] denotes the real part of the quantity in square brackets. We assume that the geopotential perturbation in the skirt is dominated by that of the core mode (27). Then, upon integrating (21), we obtain

$$q_{\rm s}^{(m,n)}(r,t) = -\frac{in}{r} \frac{d\bar{\eta}^{-1}}{dr} \Phi(r) {\rm e}^{-in\bar{\Omega}t} \int_0^t {\rm d}t' a(t') {\rm e}^{i(n\bar{\Omega}-\omega)t'}.$$
(36)

Substituting (33) and (35) into (32), and using (36) to evaluate (35), we obtain the following:

$$\frac{\mathrm{d}}{\mathrm{d}t}|a|^2 = -\frac{2n^2}{\langle Q, Q \rangle} \int_{r_v}^{\infty} \mathrm{d}r \frac{\mathrm{d}\bar{\eta}^{-1}}{\mathrm{d}r} \Phi \Phi^* Re\left[\int_0^t \mathrm{d}t' a^*(t)a(t')\mathrm{e}^{i(n\bar{\Omega}-\omega)(t'-t)}\right].$$
(37)

To simplify the analysis, we let  $a(t') \simeq a(t)$  in the time integral. Then,

$$Re\left[\int_{0}^{t} \mathrm{d}t' a^{*}(t)a(t')e^{i(n\bar{\Omega}-\omega)(t'-t)}\right] \simeq |a(t)|^{2} \frac{\sin\left[(n\bar{\Omega}-\omega)t\right]}{n\bar{\Omega}-\omega} \simeq |a(t)|^{2} \frac{\pi\delta(r-r_{*})}{n|\mathrm{d}\bar{\Omega}/\mathrm{d}r|}.$$
(38)

The second approximation in (38) assumes that  $t \gg \omega^{-1}$ . Substituting (38) into (37) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}|a| = \gamma |a|,\tag{39}$$



Fig. 3. The complex frequency v of a discrete vortex-Rossby-wave is found by solving the mode equation (41) along a Landau contour (b). The mode equation along contour (a) has no regular solution. In both (a) and (b), the asterisk marks the point where  $n\bar{\Omega}(r) = v$ .

in which

$$\gamma = -\frac{\pi n}{\langle Q, Q \rangle} \left. \frac{\mathrm{d}\bar{\eta}^{-1}/\mathrm{d}r \ \Phi \Phi^*}{\left|\mathrm{d}\bar{Q}/\mathrm{d}r\right|} \right|_{r_*}.$$
(40)

Eq. (39) has the solution  $a(t) \propto e^{\gamma t}$ . As explained previously, a wave with positive norm decays ( $\gamma < 0$ ) if  $d\bar{\eta}^{-1}/dr|_{r_*}$  is positive, and grows ( $\gamma > 0$ ) if  $d\bar{\eta}^{-1}/dr|_{r_*}$  is negative. If  $d\bar{\eta}^{-1}/dr|_{r_*}$  is zero, then the wave is neutral. Although we derived (39) under the implicit assumption that  $t \leq |\gamma|^{-1}$ , it can remain valid for much longer times (SMR).

Eq. (40) is of theoretical value, because it shows that the evolution of a discrete vortex-Rossby-wave is eventually controlled by the resonance at  $r_*$ . However, the practical value of (40) is limited, because it requires prior knowledge of  $\Phi$  and  $r_*$ . In some cases, these are easily approximated (Section 4.1); more generally they are not.

An alternative and often more convenient method for computing the frequency and decay rate of a core vortex-Rossby-wave is Landau's method.<sup>3</sup>Landau's method is to solve the following eigenmode problem:

$$\begin{bmatrix} \frac{1}{r}\frac{\partial}{\partial r}\frac{r}{\bar{\eta}\bar{\xi}}\frac{\partial}{\partial r} - \frac{S(r)n^2}{\bar{\eta}\bar{\xi}r^2} - \frac{1}{f^2l_{\rm R}^2} + \frac{1}{r}\frac{d\bar{\eta}^{-1}}{dr}\frac{n}{n\bar{\Omega}-\nu} \end{bmatrix}_L \Phi = 0,$$
  
$$\Phi(0) = \Phi(\infty) = 0.$$
 (41)

The complex eigenvalue,  $v \equiv \omega + i\gamma$ , gives the oscillation frequency ( $\omega$ ) and exponential decay rate ( $\gamma$ ) of the core wave. In (41), the subscript *L* indicates that *r* is a contour in the complex plane that arcs over the point where  $n\bar{\Omega}(r) = v$  (see Fig. 3). If the contour of integration were taken along the real *r*-axis, and if  $\bar{\eta}$  were monotonic, there would be no discrete eigenvalues.<sup>4</sup> Landau's method derives from a Laplace transform solution to the initial-value problem. It has been used previously to obtain the complex frequencies of discrete vortex-Rossby-waves in two-dimensional vortices (BDL; Corngold, 1995; Spencer and Rasband, 1997; Schecter et al., 2000) as well as three-dimensional quasigeostrophic

<sup>&</sup>lt;sup>3</sup> Landau (1946) used a similar technique to compute the complex frequency of a collisionless plasma oscillation.

<sup>&</sup>lt;sup>4</sup> This is why damped discrete vortex-Rossby-waves are properly called "quasi-modes."

vortices (SMR). In principle, the decay rate  $\gamma$  obtained by Landau's method is more accurate than (40), which formally requires  $\gamma/\omega \ll 1$ . It is proved elsewhere (e.g. SMR) that the Landau solution for  $\gamma$  converges to (40) as  $\gamma/\omega$  approaches zero.<sup>5</sup>

#### 4. Comparison of AB theory to a primitive equation model

This section presents numerous simulations of vertical alignment and horizontal axisymmetrization, governed by the conservative linearized primitive equations (7)–(10). In each simulation, the vortex is initialized with a potential vorticity perturbation of the form

$$\Pi(r,\lambda,z,t=0) = cr^{n-1}\frac{\mathrm{d}\Pi}{\mathrm{d}r}(r)\cos\left(\frac{m\pi z}{H}\right)\cos\left(n\lambda\right),\tag{42}$$

in which  $\overline{\Pi} = N^2 \overline{\eta}$  and c is a constant. A similar perturbation may arise from an impulse of environmental shear. Appendix C explains how u, v and  $\phi$  are initialized in a manner that is consistent with (42), and how the tendency equations are subsequently integrated.

After a few eddy turn-overs, a discrete vortex-Rossby-wave tends to dominate the perturbation. In all cases considered, the basic state vorticity varies monotonically with radius. As a result, in AB theory, the wave is either damped or neutral (Montgomery and Shapiro, 1995; Ren, 1999). We will compare the decay rate of the wave to AB theory.

We will present our numerical results in natural units. Time will be measured in units of central rotation periods,  $2\pi/\bar{\Omega}(0)$ , and lengths will be measured in units of the core radius,  $r_v$ . Vortices with the same  $\bar{\zeta}(r/r_v)/\bar{\zeta}(0)$ ,  $R_o$ , and  $l_R/r_v$  obey equivalent dynamics in natural units.

# 4.1. Rankine-with-skirt vortex

We will first examine the symmetrization of a Rankine-with-skirt (RWS) vortex. An ideal RWS vortex is discontinuous, and defined by

$$\zeta(r) = Z_0 \Theta(r_v - r) + Z_0 \Delta \zeta_s(r) \Theta(r - r_v), \qquad (43)$$

in which  $Z_0$  is a constant frequency,  $\Theta$  is the Heaviside step-function,  $\zeta_s$  is of order unity or less for all r, and  $\zeta_s(r_v) = 1$ . The first term on the right-hand-side of (43) represents a Rankine core of radius  $r_v$ , and the second term represents a skirt. The skirt has negligible circulation; therefore,  $\Delta \ll 1$ . The unperturbed angular rotation frequency is given approximately by (25), and precisely by  $\overline{\Omega} = r^{-2} \int_0^r dr' r' \overline{\zeta}$ .

For the primitive equation simulations, we will use the following continuous approximation of (43):

$$\bar{\zeta}(r) = \frac{Z_{o}}{\mu} \left[ 1 - b \tanh\left(\frac{r - r_{v}}{\delta r}\right) \right] \Theta(r_{z} - r) + Z_{o} \Delta \frac{\zeta_{s}(r)}{2} \left[ 1 - \tanh\left(\frac{r_{v} - r}{\delta r}\right) \right], \tag{44}$$

<sup>&</sup>lt;sup>5</sup> In SMR, convergence is proved for the quasigeostrophic limit,  $R_0 \rightarrow 0$ . However, the proof is easily generalized to finite  $R_0$ , in the context of AB theory.

in which  $\mu = 2.01$ , b = 1.01,  $\delta r = r_v/30$ ,  $r_z = 1.09r_v$ . By construction, the prefactor of the step-function vanishes at  $r_z$ , thereby ensuring continuity. We will consider the specific case in which

$$\zeta_{s}(r) = \begin{cases} 2 - \frac{r}{r_{v}} & r < 2r_{v}, \\ 0 & r > 2r_{v}. \end{cases}$$
(45)

Fig. 6a shows the relative vorticity and angular rotation frequency of the continuous RWS vortex, with  $\Delta = 0.0249$ .

#### 4.1.1. AB theory for n = 1 and n = 2 discrete vortex-Rossby-waves

In Appendix B, we analyze the resonantly damped vortex-Rossby-waves of an RWS vortex. There, we show that the frequencies, critical radii and decay rates of such waves are approximately given by

$$\omega = Z_{\rm o} \left[ \frac{n}{2} + \frac{n}{(1+R_{\rm o})(1+R_{\rm o}\Delta)} G_{mn}^{\rm (RWS)}(r_v, r_v) \right],\tag{46}$$

$$r_* = r_v \left[ 1 + \frac{2}{(1+R_0)(1+R_0\Delta)} G_{mn}^{(\text{RWS})}(r_v, r_v) \right]^{-1/2},$$
(47)

$$\gamma = -\frac{\pi n f^2}{(1+R_0)(1+R_0\Delta)} \frac{r_*^3}{r_v^2} \left[ G_{mn}^{(\text{RWS})}(r_*, r_v) \right]^2 \frac{\mathrm{d}\bar{\eta}^{-1}}{\mathrm{d}r}(r_*), \tag{48}$$

in which  $G_{mn}^{(\text{RWS})}$  is the Green function of an RWS vortex. The geopotential wave-forms  $(\Phi)$  are approximately proportional to  $G_{mn}^{(\text{RWS})}(r, r_v)$ .

We begin by considering the n = 1 discrete vortex-Rossby-waves of an RWS vortex, with  $\zeta_s$  given by (45), and  $\Delta = 0.0249$ . Fig. 4a shows the critical radius (47) versus the Rossby number, for several values of  $l_R$  between  $0.5r_v$  and  $1.45r_v$ . The solid curves are for waves in cyclones ( $R_o > 0$ ) whereas the dashed curves are for waves in anticyclones ( $R_o < 0$ ). As  $|R_o|$  increases along each curve toward 0.1,  $r_*$  remains approximately constant. As  $|R_o|$  increases past 0.1,  $r_*$  decays toward  $r_v$ . For each anticyclonic curve,  $r_*$  exhibits a slight growth with  $|R_o|$ , before decay.

Fig. 4b shows the decay rate (48) versus Rossby number, for the same values of  $l_{\rm R}$ . The plotted decay rate is scaled to  $|\bar{\zeta}(0)|$ . QG theory predicts that the scaled decay rate is constant for a given  $l_{\rm R}$ , and equal to the  $|R_0| \rightarrow 0$  limit of AB theory. Fig. 4b indicates that AB theory can diverge appreciably from QG theory as the magnitude of the Rossby number approaches and exceeds unity. Such divergence occurs most dramatically for the case in which  $l_{\rm R} = 1.45r_v$ . For this case, QG theory predicts that the wave is neutral. The wave is neutral because its critical radius is greater than twice  $r_v$ , where the basic state potential vorticity gradient is zero. In AB theory, increasing  $|R_0|$  to order unity decreases the critical radius below twice  $r_v$ , where the potential vorticity gradient is finite; accordingly, the wave becomes damped.

Fig. 5 illustrates how the n = 2 discrete vortex-Rossby-waves vary with the Rossby number and internal deformation radius. As before, the solid curves are for cyclones, and the dashed curves are for anticyclones. For all values of  $l_{\rm R}$  and the Rossby number, the



Fig. 4. The n = 1 discrete vortex-Rossby-waves of an RWS vortex ( $\Delta = 0.0249$ ). (a) Critical radius vs. Rossby number. (b) Decay rate vs. Rossby number. Solid/dashed curves are for cyclones/anticyclones.

critical radius is less than twice  $r_v$ . Otherwise, the n = 2 and n = 1 waves are similar. For example, the scaled decay rate generally decreases as the magnitude of the Rossby number increases, and as  $l_R$  decreases.

We have purposely not presented anticyclonic data with  $R_o \leq -1$ . In this region of parameter space, there are radii at which  $\bar{\eta}\bar{\xi} \to 0$ , and thus  $\mathcal{D}_I^2 \to \infty$ ; consequently, AB theory is formally invalid. Furthermore, the plotted AB theory becomes increasingly unreliable as  $R_o \to -1$ . This formal break-down of AB theory coincides with a symmetric (centrifugal) instability of the anticyclone at Rossby numbers less than minus one (e.g. Smyth and McWilliams, 1998).

#### 4.1.2. Vertical alignment

We now examine the evolution of an n = 1 perturbation (42) that is governed by the linearized, conservative primitive equations. The vertical wave-number *m* of this perturbation is arbitrary, insofar as  $l_{\rm R} = NH/m\pi |f| = 1.45r_v$ . If m = 1, the perturbation corresponds to a simple tilt. Increasing *m* increases the number of nodes in the misalignment.

Fig. 6b shows the evolution of the geopotential perturbation  $\phi$ , at a fixed height, for the case in which the Rossby number is 2. Evidently, the geopotential perturbation behaves like a single azimuthally propagating wave. The decay of the wave corresponds to vertical alignment.



Fig. 5. The n = 2 discrete vortex-Rossby-waves of an RWS vortex ( $\Delta = 0.0249$ ). (a) Critical radius vs. Rossby number. (b) Decay rate vs. Rossby number. Solid/dashed curves are for cyclones/anticyclones.

Fig. 6c plots the wave-amplitude versus time for cases in which the Rossby numbers are 2 and 0.25. Here, the wave-amplitude is adequately measured by a local "tilt" variable,  $|\phi^{(m,1)}(r_v, t)|$ . The solid curves are from the primitive equation model. The dashed curves are the AB theory of resonant damping; i.e.  $e^{\gamma t}$ , in which  $\gamma$  is given by (48). As predicted by AB theory, there is a transition from undamped to damped waves as the Rossby number increases to order unity. Furthermore, AB theory adequately predicts the damping rate at a relatively large Rossby number (two), where QG theory does not formally apply.

It is no surprise that AB theory compares favorably to the primitive equation model for these examples. Fig. 6d shows that  $\mathcal{D}_I^2 \ll 1$  for both cases in Fig. 6c; hence, the AB approximation is justified. Here, we have used the formula  $\mathcal{D}_I^2 = (\omega - n\bar{\Omega})^2/\bar{\eta}\bar{\xi}$ , in which  $\omega$  is given by (46). Shortly, we will address the qualitative deficiencies of the AB approximation (and balance models in general) for some cases in which  $\mathcal{D}_I^2 \gtrsim 1$ .

#### 4.1.3. Horizontal axisymmetrization

We now examine the evolution of an n = 2 perturbation (42) that is governed by the linearized, conservative primitive equations. This perturbation corresponds to an elliptical distortion of the horizontal flow. If m = 1, the major axis of the ellipse rotates 90° across



Fig. 6. The vertical alignment of an RWS vortex;  $\Delta = 0.0249$ ;  $l_{\rm R} = 1.45r_v$ . (a) Basic state relative vorticity, and angular rotation frequency, normalized to  $\bar{\zeta}(0)$ . (b) Geopotential perturbation  $\phi$  in the horizontal plane, at an arbitrary height. The decay of  $\phi$  corresponds to vertical alignment. The evolution was obtained from a numerical integration of the linearized primitive equations. Time *t* is in units of  $2\pi/\bar{\Omega}(0)$ .  $R_0 = 2.0$ . (c) The tilt of the vortex vs. time, for two different Rossby numbers. Solid curves: primitive equation model. Dashed curves: AB theory for resonantly damped n = 1 vortex-Rossby-waves. (d)  $\mathcal{D}_I^2$  vs. radius, for the two cases in (c).

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the horizontal plane at z = H/2. Increasing *m* increases the number of nodes of ellipticity between z = 0 and z = H.

Fig. 7a shows the evolution of the geopotential perturbation  $\phi$ , at a fixed height, for the case in which the Rossby number is unity, and  $l_{\rm R} = NH/m\pi |f| = 1.45r_v$ . As before, the geopotential perturbation behaves like a single azimuthally propagating wave. Here, the decay of the wave corresponds to horizontal axisymmetrization.

Fig. 7b plots the wave-amplitude versus time, for two cases in which  $R_0 = 1$ , and  $l_R/r_v$  is either 0.5 or 1.45. The wave-amplitude is adequately measured by a local "ellipticity" variable,  $|\phi^{(m,2)}(r_v, t)|$ . The solid and dashed curves have the same significance as in Fig. 6c. For all cases, the AB theory of resonant damping accurately describes the decay of ellipticity. Once again, good agreement between the primitive equation model and AB theory is in accord with the fact that  $\mathcal{D}_I^2 \ll 1$  (Fig. 7c).

# 4.1.4. Limitation of balance

The applicability condition for the present asymmetric balance model,  $D_I^2 \ll 1$ , can fail if the Rossby number is too large. For example, consider an n = 2 discrete vortex-Rossbywave in a Rankine cyclone. Fig. 8a plots the AB theory of  $D_I^2$  versus radius for this perturbation, for the case in which  $l_R$  equals the core radius  $r_v$ . Each curve in Fig. 8a corresponds to a distinct Rossby number, between 1 and 10. Below the critical radius,  $D_I^2$  is much less than unity. Beyond the critical radius,  $D_I^2$  is greater than unity, if the Rossby number exceeds one.

According to AB theory, all discrete vortex-Rossby-waves of a Rankine cyclone are neutral. Fig. 8b compares this prediction to a set of numerical experiments, with  $l_{\rm R}$  equal to  $r_v$ . The experiments are based on the primitive equation model. The Rankine cyclone is approximated by (44), with  $\Delta$  equal to zero. The initial potential vorticity perturbation is given by (42), with n = 2. At large Rossby numbers, a *growing* discrete vortex-Rossby-wave dominates the perturbation, contradicting AB theory. The growth rate  $\gamma$  is obtained from an exponential fit of ellipticity versus time. The scaled growth rates are shown in Fig. 7b, for Rossby numbers between 1 and 20. As the Rossby number increases, beyond 2, the *e*-folding time approaches about 7 central rotation periods.

The observed growing modes are analogous to those studied by Ford (1994), for the case of a shallow-water vortex on the f-plane. Ford attributes the instability to a resonant interaction between the core vortex-Rossby-wave, and an outward-propagating, spiral, inertia-buoyancy wave. For a comprehensive discussion of linear instability due to wave-wave resonances, we refer the reader to Sakai (1989).

Fig. 8c shows the geopotential perturbation of the unstable modes for Rossby numbers equal to 3, 5 and 10. The azimuthal phase-velocity of each mode is between 12 and 13% smaller than the phase-velocity of its neutral counterpart in the AB model. In the skirt, the geopotential perturbation consists of spiral bands. The radial wave-length of the spiral pattern decreases as the Rossby number increases. Such behavior is easily derived.

To begin with, we may express the eigenmode problem of the primitive equation model as a single equation for the geopotential perturbation,  $\phi \propto \Phi(r) \exp[i(n\lambda - \nu t)] + \text{c.c.}$  At large radii, where  $\overline{\Omega}, \overline{\zeta} \rightarrow 0$ , this eigenmode equation reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\Phi}{\partial r} - \frac{\omega^2/f^2 - 1}{l_{\rm R}^2}\Phi = 0,$$
(49)



Fig. 7. The horizontal axisymmetrization of an RWS vortex;  $\Delta = 0.0249$ ;  $R_0 = 1$ . (a) Geopotential perturbation  $\phi$  in the horizontal plane, at an arbitrary height. As  $\phi$  decays, elliptical flow becomes circular in the horizontal plane. The evolution was obtained from a numerical integration of the linearized primitive equations. Time *t* is in units of  $2\pi/\bar{\Omega}(0)$ .  $l_R/r_v = 1.45$ . (b) The ellipticity of the vortex vs. time, for two different internal deformation radii. Solid curves: primitive equation model. Dashed curves: AB theory for resonantly damped n = 2 vortex-Rossby-waves. (c)  $D_I^2$  vs. *r*, for the two cases in (b).



Fig. 8. The n = 2,  $l_R = r_v$ , Rossby-inertia-buoyancy (RIB) instability. (a)  $\mathcal{D}_I^2$  vs. radius, for several large Rossby numbers. (b) The growth rate of the RIB wave in a Rankine vortex. Diamonds: primitive equation model. Dotted-line: AB theory. (c) The geopotential perturbation of the unstable RIB wave, at an arbitrary height, for several Rossby numbers. The dashed circle is the domain boundary of the primitive equation model, where a sponge ring absorbs outward propagating waves. (d) The growth rate of the RIB wave in an RWS vortex, with  $R_0 = 3$ . Diamonds: primitive equation model. Squares: approximate AB theory (48). Increasing the slope of the skirt ( $\Delta$ ) suppresses the instability, via enhanced resonant damping.

In general, the eigenfrequency  $\nu$  is complex; however, the observed instability is weak. Accordingly, we have approximated  $\nu$  by its real part  $\omega$ . For an outward propagating wave, the proper solution to (49) is a Hankel function of the first kind:  $\Phi = cH_0^{(1)}(\rho) \sim c\sqrt{2/\pi\rho} \exp[i(\rho - \pi/4)]$ , in which *c* is a constant and  $\rho = \sqrt{\omega^2/f^2 - 1} r/l_{\rm R}$ . From this solution, the radial wave-length of the spiral is simply

$$l_{\rm s} = \frac{2\pi l_{\rm R}}{\sqrt{\omega^2/f^2 - 1}}.$$
(50)

Clearly,  $l_s$  decreases as the Rossby number ( $\sim \omega/f > 1$ ) increases. Fig. 8c provides a favorable comparison of  $l_s$  to the observed radial wave-lengths.

By increasing  $d\bar{\eta}^{-1}/dr|_{r_*}$  above zero, it is possible to suppress the RIB instability. For the basic state given by (44),  $d\bar{\eta}^{-1}/dr|_{r_*}$  has the same sign as, and is approximately proportional to  $\Delta$ . Fig. 8d shows  $\gamma$  versus  $\Delta$ , for the case in which the Rossby number is 3, and the internal deformation radius is  $r_v$ . As before, the diamonds were obtained from exponential fits of ellipticity versus time in the primitive equation model. The squares correspond to AB theory (48). Fig. 8d clearly indicates that  $\gamma$  is made negative by increasing  $\Delta$ . For large values of  $\Delta$ , resonant damping dominates the RIB instability, and AB theory provides a good approximation for  $\gamma$ .

#### 4.2. Gaussian vortex

The RWS vortex has didactic value, and may be a reasonable model for the polar stratospheric vortex. However, the symmetrization rate of a smoother potential vorticity distribution can exhibit opposite tendencies with  $R_0$  and  $l_R$ . As an example, we consider a Gaussian vortex:

$$\bar{\zeta} = Z_0 e^{-r^2/2\bar{r}^2}.$$
 (51)

Fig. 10a plots the basic state vorticity and angular rotation frequency of a Gaussian vortex.

For brevity, we will restrict our attention to vertical alignment, which occurs by the decay of the n = 1 Fourier components of the perturbation. Fig. 9 illustrates the properties of the n = 1 discrete vortex-Rossby-waves. The complex wave frequencies were obtained by Landau's method (41). Fig. 9a shows the critical radius  $r_*$  versus the Rossby number  $R_0$ , for several values of  $l_R$  between  $0.375r_v$  and  $r_v$ ; here,  $r_v \equiv 3\overline{r}$ . To some extent, the curves are qualitatively similar to the RWS results (Figs. 4a and 5a). However, the ascent of  $r_*$  at large (positive) Rossby numbers is distinct.

Fig. 9b shows the scaled decay rate,  $-\gamma/|\bar{\zeta}(0)|$ , versus Rossby number  $R_0$ , for the same values of  $l_R$ . Along the cyclonic curves (solid), the scaled decay rate increases with  $|R_0|$ . Along the anticyclonic curves (dashed), the scaled decay rate decreases, and then sharply increases as  $|R_0|$  approaches one. Note also that the decay rate increases as  $l_R$  decreases, *in contrast* to the RWS results.

We now examine the evolution of an n = 1 perturbation (42) that is governed by the linearized, conservative primitive equations. The vertical wave-number *m* of this perturbation is arbitrary, insofar as  $l_{\rm R} = NH/m\pi |f| = 0.75r_v$ . Recall that if m = 1, the perturbation corresponds to a simple tilt. Increasing *m* increases the number of nodes in the misalignment.



Fig. 9. The n = 1 discrete vortex-Rossby-waves of a Gaussian vortex. (a) Critical radius vs. Rossby number. (b) Decay rate vs. Rossby number. Solid/dashed curves are for cyclones/anticyclones.

Fig. 10b shows the evolution of the geopotential perturbation  $\phi$ , at a fixed height, for the case in which the Rossby number is 2. The geopotential perturbation behaves like a single azimuthally propagating wave. The phase of the wave varies slightly with radius. The temporal decay of the wave corresponds to vertical alignment.

Fig. 10c plots the wave-amplitude versus time for cases in which the Rossby numbers are 2, 0.25, and -0.25. The wave-amplitude is measured by a local "tilt" variable,  $|\phi^{(m,1)}(\sqrt{2}\bar{r}, t)|$ . The solid curves are from the primitive equation model. The dashed curves are from the AB theory of resonant damping; i.e.  $e^{\gamma t}$ , in which  $\gamma$  is obtained from Landau's method. The decay rate predicted by AB theory is adequate for all three cases; however, it is least accurate (+28%) for the case in which the Rossby number is 2. Increasing error with Rossby number coincides with increasing  $\mathcal{D}_I^2$  (Fig. 10d).

## 4.3. Hurricane Olivia

To this point, we have considered modest geophysical vortices, that have Rossby numbers of order ten or less, and  $l_R/r_v$  of order unity or less. In contrast, full-strength hurricanes have Rossby numbers of order 100, and  $l_R/r_v$  of order ten (for  $m \leq 4$ ). In this subsection,



Fig. 10. The vertical alignment of a Gaussian vortex:  $l_R = 0.75r_v$ . (a) Basic state relative vorticity, and angular rotation frequency, normalized to  $\bar{\zeta}(0)$ . (b) Geopotential perturbation  $\phi$  in the horizontal plane, at an arbitrary height. The decay of  $\phi$  corresponds to vertical alignment. The evolution was obtained from a numerical integration of the linearized primitive equations. Time *t* is in units of  $2\pi/\bar{\Omega}(0)$ .  $R_0 = 2.0$ . (c) The tilt of the vortex vs. time, for three different Rossby numbers. Solid curves: primitive equation model. Dashed curves: AB theory for resonantly damped n = 1 vortex-Rossby-waves. (d)  $\mathcal{D}_1^2$  vs. radius, for the three cases in (c).

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we consider the alignment of a barotropic vortex with hurricane-like parameters. We note that actual hurricanes may exhibit more complex behavior, in part because they are baroclinic and contain secondary convection that operates on a time scale comparable to that of symmetrization.

Fig. 11a shows the relative vorticity and angular rotation frequency of a monotonic barotropic vortex that resembles Hurricane Olivia (Reasor et al., 2000). In this plot,  $\bar{\zeta}$  and  $\bar{\Omega}$  are normalized to  $Z_0 = 8.84 \times 10^{-3} \text{ s}^{-1}$ , and r is in units of  $r_v = 30 \text{ km}$ . At  $15^{\circ}$ N, the Rossby number  $R_0$  of Olivia is 234.18. Furthermore, using  $N = 1.21 \times 10^{-2} \text{ s}^{-1}$ ,  $H \simeq 10 \text{ km}$  and m = 1, we obtain  $l_{\text{R}}/r_v = 34.29$ . Note that we have artificially filled the eye (r < 8 km) of Olivia with cyclonic vorticity. In doing so, we have removed possible inflection-point instabilities (e.g. Michalke and Timme, 1967; Schubert et al., 1999) that could otherwise interfere with the mechanics of linear symmetrization.

The specific functional form of  $\overline{\xi}$  is given below

$$\bar{\zeta} = Z_{0} \frac{1 + \tanh\left[(2r_{v} - r)/\delta r\right]}{2b_{1}} \left[ e^{-(r/\sigma_{1})^{2}} \left(b_{2} + b_{3} \frac{r^{2}}{r_{v}^{2}}\right) + b_{4} e^{-(r/\sigma_{2})^{2}} \right] + Z_{0} \frac{1 - \tanh\left[(2r_{v} - r)/\delta r\right]}{2b_{5}} \left(\frac{r}{r_{v}}\right)^{p},$$
(52)

in which  $\delta r = 2r_v/15$ ,  $\sigma_1 = 0.343r_v$ ,  $\sigma_2 = 1.30r_v$ ,  $b_1 = 1.36$ ,  $b_2 = 1.145$ ,  $b_3 = 10.92$ ,  $b_4 = 0.218$ ,  $b_5 = 23.683$ , and p = -1.783. As usual, the angular rotation frequency is obtained from the relation  $\bar{\Omega}(r) = r^{-2} \int_0^r dr' r' \bar{\zeta}(r')$ .

Fig. 11b shows the evolution of a geopotential perturbation that is created by a tilt of Olivia [Eq. (42); (m, n) = (1, 1)]. The data were obtained from a numerical integration of the linearized primitive equations. After about five central rotation periods (roughly 2 h), the perturbation resembles a discrete azimuthally propagating wave. As the wave decays, the vortex becomes vertically aligned. A fit to the simulation data between five and forty rotation periods yields a wave frequency  $\omega = 0.044$ , and a decay rate  $\gamma = -5.5 \times 10^{-3}$ , in units of  $\overline{\zeta}(0)$ . Landau's method provides the corresponding AB value of the complex mode frequency:  $\omega = 0.048$  and  $\gamma = -3.8 \times 10^{-3}$ .

Fig. 11c shows the tilt of the vortex,  $|\phi^{(m,1)}(0.567r_v, t)|$ , as a function of time. The solid curve represents the primitive equation simulation. The long-dashed curve represents a numerical integration of the AB model (15) and (16).<sup>6</sup> The short-dashed curve represents exponential decay, using the Landau value for  $\gamma$ . We conclude that AB theory provides a good approximation for linear alignment. It differs slightly from the primitive equation model, because  $\mathcal{D}_I^2 \ll 1$  is only marginally satisfied in the core (Fig. 11d).

Despite the large Rossby number ( $R_0 = 234.18$ ) and "Froude number" ( $R_0 r_v / l_R = 6.8$ ), Fig. 11 shows no trace of an n = 1 RIB instability. The same holds for a similar n = 2 simulation. So, at least for low wave-numbers, it would appear that the symmetrization processes accounted for by AB theory dominate conceivable unbalanced instabilities of a hurricane-like vortex. This result was anticipated by Montgomery and Shapiro (1995).

<sup>&</sup>lt;sup>6</sup> The initial condition of the AB simulation is given by (42), with  $\Pi \rightarrow q$  on the left-hand-side.



Fig. 11. The vertical alignment of a barotropic vortex resembling Hurricane Olivia (1994):  $R_0 = 234.18$ ,  $l_R/r_v = 34.29$ . (a) Basic state. *Xs* : relative vorticity (700mb) from airborne dual doppler velocimetry of Olivia, as described in Reasor et al. (2000). Solid curve: vorticity fit (52). Dashed curve: angular velocity from vorticity fit. Vorticity and angular velocity are normalized to  $\bar{\zeta}(0)$ . (b) Geopotential perturbation  $\phi$  in the horizontal plane, at an arbitrary height. The decay of  $\phi$  corresponds to vertical alignment. The evolution was obtained from a numerical integration of the linearized primitive equations. Time *t* is in units of  $2\pi/\bar{\Omega}(0)$ . (c) The tilt of the vortex vs. time. Solid curve: primitive equation model. Long-dashed curve: AB model. Short-dashed curve: AB theory for resonantly damped n = 1 vortex-Rossby-wave. (d)  $\mathcal{D}_I^2$  vs. radius for wave in (b) and (c).

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# 5. Summary and discussion

We have presented a linear AB theory of discrete vortex-Rossby-waves in a barotropic mean vortex. If the vortex has a monotonic potential vorticity profile, such waves are damped by a resonance with the fluid rotation frequency at a critical radius  $r_*$ . An analytical expression (40) was derived for the decay rate of the wave-amplitude. According to this formula, the decay rate is proportional to the value of  $d\bar{\eta}^{-1}/dr = -N^2\bar{\Pi}^{-2} d\bar{\Pi}/dr$  at  $r_*$ . For Rossby numbers of order unity or greater, there are cases in which the AB and QG formulas for the decay rate can differ by orders of magnitude. For such cases, AB theory proved superior upon comparison to numerical integrations of the primitive equations (Figs. 6c and 10c).

The results of this article may help explain why some geophysical vortices are more robust than others when exposed to environmental shear. Some recent work has studied the adverse effect of vertically-sheared horizontal winds on tropical cyclone development. Fig. 12 illustrates how vertical shear can weaken a tropical cyclone. These data are from a nonhydrostatic, moist numerical simulation by Frank and Ritchie (2001; Fig. 5). The simulation is initialized with an axisymmetric tropical depression. In time the central pressure drops, and the maximal wind grows to hurricane intensity. After 48 h, the environmental flow acquires vertical shear. If the shear-rate is sufficiently large, as in this example, the vortex tilts. The figure suggests that the shear-induced tilt contributes to decay of the central pressure anomaly, and thereby to weakening of the maximal wind.

In Section 4, we showed that tilts are often supported by n = 1 (m > 0) discrete vortex-Rossby-waves, which are inclined to decay by resonant damping. Consider two cyclones, I and II, which have identical cores, but differing skirts. Specifically, the skirt



Fig. 12. Weather prediction models demonstrate that tropical cyclones intensify/weaken in the absence/presence of vertically sheared environmental wind. As the shear-induced tilt increases, the central pressure anomaly (DP = initial minus current central pressure) decreases. Here, tilt is defined as the distance between the centers of rotation at 850 and 200 mb (long-dashed), or 850 and 500 mb (short-dashed). The large "tilt" shown after t = 72 h is thought to be an artificial result of the vortex losing a distinct center in its upper levels. This figure is adapted from Frank and Ritchie (2001).



Fig. 13. The axisymmetrization of an electron vortex, governed approximately by the 2D Euler equations. (a) Evolution of the vorticity field  $\zeta_{tot}$ . (b) Evolution of the quadrupole moment (ellipticity) of  $\zeta_{tot}$ . Time is in units of  $2\pi/\bar{\Omega}(0)$ . This figure is adapted from Scheeter et al. (2000).

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potential vorticity of cyclone I has a larger negative slope. AB theory suggests that the vortex-Rossby-waves will damp faster on cyclone I. It stands to reason that cyclone I is better able than cyclone II to resist tilting, and thereby weakening, under vertical shear. Accordingly, we speculate that cyclone I requires a larger vertical shear-rate than cyclone II to nullify its intensification.

Before investigating the role of discrete vortex-Rossby-waves in regulating the strength of vortices that are exposed to external shear, several basic issues should be addressed. To begin with, resonant damping generally competes with growth mechanisms. For example, at large Rossby numbers, a vortex-Rossby-wave can grow by coupling to a spiral inertia-buoyancy wave (Section 4.1.4; Ford, 1994). We have shown that growth by such coupling yields to resonant damping, if  $d\bar{\Pi}/dr|_{r_*}$  is sufficiently large, and of the proper sign. Future work should provide a more comprehensive analysis of this suppression.

Future work should also extend the theory of discrete vortex-Rossby-waves to baroclinic vortices, which have vertically varying rotation. Baroclinicity will clearly complicate the analysis of resonant damping, and of competing instability mechanisms. Nevertheless, there are likely special cases for which the analysis is feasible.

Furthermore, one may generalize the basic state to include an elliptical vortex column in permanent horizontal shear. Satellite images suggest that such basic states more accurately represent Jovian vortices, such as the Great Red Spot. Like circular vortices, we propose that elliptical vortices will favor vertical alignment, due to the resonant damping of discrete vortex-Rossby-waves. Such damping may also quiet internal disturbances about the elliptical horizontal flow.

Future work should also elaborate upon the nonlinear evolution of discrete vortex-Rossbywaves. The nonlinear evolution was previously studied in the context of two-dimensional Euler flow (BDL; Pillai and Gould, 1994; Schecter et al., 2000; Balmforth et al., 2001). Fig. 13 is a laboratory experiment that illustrates the nonlinear resonant damping of an n = 2 wave (Schecter et al., 2000). The working 2D Euler fluid in this experiment is a magnetized pure electron plasma. Initially, the wave decays at a rate given by the linear theory of resonant damping (dashed line). However, as vorticity wraps into a cat's eyes pattern at the critical radius  $r_*$ , the wave amplitude bounces and then equilibrates. The theoretical value for the bounce period  $\tau_b$  is proportional to one over the square root of the wave amplitude. As indicated by the figure, the theoretical value of  $\tau_b$  (extended tick mark) is in good agreement with the experiment. If the initial wave amplitude is below a threshold, nonlinear bouncing will not occur, and the wave amplitude will decay toward zero (Balmforth et al., 2001). Presumably the threedimensional discrete vortex-Rossby-waves examined here exhibit analogous nonlinear behavior.

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# Appendix A

In this appendix, we describe a general method for constructing the Green function  $G_{mn}(r, r')$ , which relates the geopotential perturbation to the pseudo potential vorticity perturbation in the asymmetric balance model. The Green function must satisfy a homogeneous differential equation,

$$L[G_{mn}] \equiv f^2 \left( \frac{1}{r} \frac{\partial}{\partial r} \frac{r}{\bar{\eta}\bar{\xi}} \frac{\partial}{\partial r} - \frac{S(r)n^2}{\bar{\eta}\bar{\xi}r^2} - \frac{1}{f^2 l_{\rm R}^2} \right) G_{mn} = 0, \tag{A.1}$$

for  $r \neq r'$ , and the jump conditions

$$\lim_{\epsilon \to 0^+} G_{mn} \Big|_{r'-\epsilon}^{r'+\epsilon} = 0, \qquad \lim_{\epsilon \to 0^+} \left. \frac{rf^2}{\bar{\eta}\bar{\xi}} \frac{\mathrm{d}G_{mn}}{\mathrm{d}r} \right|_{r'-\epsilon}^{r'+\epsilon} = 1.$$
(A.2)

Let  $f_1(r)$  and  $f_2(r)$  be solutions to  $L[f_i] = 0$ , with the boundary conditions  $f_1(0) = 0$  and  $f_2(\infty) = 0$ . The Green function is proportional to  $f_1$  for r < r', whereas it is proportional to  $f_2$  for r > r'. Provided that  $\bar{\eta}\bar{\xi}$  is continuous at r = r', the jump conditions at r' yield

$$G_{mn}(r,r') = \frac{f_1(r_<)f_2(r_>)}{W},$$
(A.3)

in which  $r_{<}(r_{>})$  is the lesser (greater) of r and r', and

$$W = \frac{r'f^2}{\bar{\eta}(r')\bar{\xi}(r')} \left[ f_1(r')\frac{df_2}{dr}(r') - \frac{df_1}{dr}(r')f_2(r') \right].$$
 (A.4)

It can be shown that W is a constant function of r'.

As *r* approaches zero, the differential equation for  $f_1$  is given by

$$\frac{d^2 f_1}{dr^2} + \frac{1}{r} \frac{df_1}{dr} - \frac{n^2}{r^2} f_1 \sim 0.$$
(A.5)

Imposing the boundary condition,  $f_1(0) = 0$ , the solution to (A.5) is  $f_1(r) \sim r^n$ . We obtain the complete function  $f_1$  by setting  $f_1(\epsilon) = c$ ,  $df_1/dr(\epsilon) = nc/\epsilon$ , and numerically integrating  $L[f_1] = 0$  forward in r. Here, c is an arbitrary constant, and the accuracy of the solution improves as  $\epsilon$  approaches zero.

As r tends toward infinity, the functions  $\bar{\eta}\bar{\xi}/f^2$  and S(r) become unity, and the differential equation for  $f_2$  becomes

$$\frac{d^2 f_2}{dr^2} + \frac{1}{r} \frac{df_2}{dr} - \frac{1}{l_R^2} f_2 \sim 0.$$
(A.6)

Therefore,  $f_2 \sim K_0(r/l_R)$ , where  $K_0$  is a modified Bessel function. Using, the asymptotic  $(r/l_R \gg 1)$  form of  $K_0$ , we have further that  $f_2 \sim \sqrt{l_R/r} \exp[-r/l_R]$ . A numerical solution

to  $f_2$  is obtained by setting  $f_2(\epsilon^{-1}) = c$ ,  $df_2/dr(\epsilon^{-1}) = -c(1+\epsilon l_R/2)/l_R$ , and integrating  $L[f_2] = 0$  backward in r.

For the special case of a Rankine vortex (25), the function  $\bar{\eta}\bar{\xi}/f^2$  is discontinuous at  $r_v$ . Specifically,

$$\frac{\bar{\eta}\bar{\xi}}{f^2} = \begin{cases} (1+R_0)^2, & r < r_v, \\ 1+R_0\frac{r_v^2}{r^2}, & r > r_v. \end{cases}$$
(A.7)

For  $r' = r_v$ , the jump conditions yield

$$G_{mn}^{(R)}(r, r_v) = \frac{f_1(r_<)f_2(r_>)}{\tilde{W}},$$
(A.8)

where  $r_{<}(r_{>})$  is the lesser (greater) of r and  $r_{v}$ , and

$$\tilde{W} = \frac{r_v}{(1+R_o)^2} \left[ (1+R_o)f_1(r_v)\frac{\mathrm{d}f_2}{\mathrm{d}r}(r_v^+) - \frac{\mathrm{d}f_1}{\mathrm{d}r}(r_v^-)f_2(r_v) \right].$$
(A.9)

Here, the symbols  $r_v^+$  and  $r_v^-$  indicate the limits as r approaches  $r_v$  from above and below, respectively. One can easily show that for  $r \leq r_v$ ,  $f_1(r) = I_n[(1 + R_o)r/l_R]$ . Fig. 14 illustrates the global structure of  $G_{mn}^{(R)}(r, r_v)$  for several values of  $R_o$ ,  $l_R$  and n.

# Appendix B

In this appendix, we derive the properties of the discrete vortex-Rossby-waves of an RWS vortex. Our approximate theory improves, in principle, as the skirt parameter  $\Delta$  approaches zero.

From the definition of an RWS vortex (43), the local absolute and average absolute vorticities are given by

$$\frac{\bar{\eta}}{f} = \begin{cases} 1 + R_{\rm o} & r < r_v, \\ 1 + R_{\rm o} \Delta \zeta_{\rm s}(r) & r > r_v, \end{cases} \quad \frac{\bar{\xi}}{f} = \begin{cases} 1 + R_{\rm o} & r < r_v, \\ 1 + R_{\rm o}(r_v/r)^2 & r > r_v. \end{cases}$$
(B.1)

Here, we have used the approximation

$$\frac{\bar{\Omega}}{Z_0} = \frac{1}{2} \left(\frac{r_v}{r}\right)^2 + \frac{\Delta}{r^2} \int_{r_v}^r \mathrm{d}r' r' \zeta_{\mathrm{s}}(r') \simeq \frac{1}{2} \left(\frac{r_v}{r}\right)^2,\tag{B.2}$$

for  $r > r_v$ , according to the assumption  $\Delta r_v^{-2} \int_{r_v}^r dr' r' \zeta_s(r') \ll 1$ .

An integral core mode equation is derived from (21), (22) and (B.1). Ignoring two terms that are proportional to  $\Delta$ , we have

$$\frac{n\bar{\Omega} - \omega}{Z_0}Q(r) + \frac{n\delta(r - r_v)}{r(1 + R_0)(1 + R_0\Delta)} \int_0^\infty dr' r' G_{mn}^{(\text{RWS})}(r, r')Q(r') = 0.$$
(B.3)

This implies that the pseudo potential vorticity and geopotential wave-forms are given by

$$Q(r) = r_v \delta(r - r_v), \qquad \Phi(r) = f^2 r_v^2 G_{nm}^{(\text{RWS})}(r, r_v), \tag{B.4}$$



Fig. 14. The Green function  $(r' = r_v)$  of a Rankine vortex. Equivalently, the geopotential wave-form  $[\Phi(r)]$  of a discrete vortex-Rossby-wave of a Rankine vortex. (a) Variation with Rossby number. (b) Variation with internal Rossby deformation radius. (c) Variation with azimuthal wave-number.

up to a common multiplicative constant. The Green function  $G_{mn}^{(\text{RWS})}(r, r_v)$  is constructed just as  $G_{mn}^{(\text{R})}(r, r_v)$  in Appendix A. The result is

$$G_{mn}^{(\text{RWS})}(r, r_v) = \frac{f_1(r_<)f_2(r_>)}{\hat{W}},$$
(B.5)

in which  $r_{<}(r_{>})$  is the lesser (greater) of r and  $r_{v}$ , and

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$$\hat{W} = \frac{r_v}{(1+R_o)^2} \left[ \frac{1+R_o}{1+R_o\Delta} f_1(r_v) \frac{\mathrm{d}f_2}{\mathrm{d}r}(r_v^+) - \frac{\mathrm{d}f_1}{\mathrm{d}r}(r_v^-) f_2(r_v) \right].$$
(B.6)

The function  $f_1(r)$  is equal to the modified Bessel function  $I_n[(1 + R_0)r/l_R]$ , for  $r \le r_v$ . The function  $f_2(r)$  satisfies  $L[f_2] = 0$ , and the boundary condition  $f_2(\infty) = 0$ . The linear differential operator L is defined by (A.1), with ((B.1);  $r > r_v$ ) used for  $\bar{\eta}$  and  $\bar{\xi}$ . The Green function  $G_{mn}^{(\text{RWS})}$  converges to  $G_{mn}^{(\text{R})}$  (Fig. 14) as  $R_0 \Delta \to 0$ .

Upon substituting  $Q(r) = r_v \delta(r - r_v)$  into (B.3), one finds that the mode frequency is given by

$$\omega = Z_{\rm o} \left[ \frac{n}{2} + \frac{n}{(1+R_{\rm o})(1+R_{\rm o}\Delta)} G_{mn}^{\rm (RWS)}(r_v, r_v) \right].$$
(B.7)

To zero order in  $\Delta$ , the critical radius  $r_*$  satisfies  $nZ_0(r_v/r_*)^2/2 = \omega$ ; i.e.

$$r_* = r_v \left[ 1 + \frac{2}{(1+R_0)(1+R_0\Delta)} G_{mn}^{(\text{RWS})}(r_v, r_v) \right]^{-1/2}.$$
(B.8)

The decay rate is obtained by substituting (B.4) into (40):

$$\gamma = -\frac{\pi n f^2}{(1+R_0)(1+R_0\Delta)} \frac{r_*^3}{r_v^2} \left[ G_{mn}^{(\text{RWS})}(r_*, r_v) \right]^2 \frac{\mathrm{d}\bar{\eta}^{-1}}{\mathrm{d}r}(r_*).$$
(B.9)

Eqs. (B.7)–(B.9) are the same as (46)–(48) in the main text.

# Appendix C

In this appendix, we describe the primitive equation model that was used to investigate vortex symmetrization. The prognostic equations for this model are given below:

$$\begin{bmatrix} \frac{\partial}{\partial t} + in\bar{\Omega}(r) + \alpha(r) \end{bmatrix} u^{(m,n)}(r,t) - \bar{\xi}(r)v^{(m,n)}(r,t) + \frac{\partial}{\partial r}\phi^{(m,n)}(r,t) = 0,$$
  
$$\begin{bmatrix} \frac{\partial}{\partial t} + in\bar{\Omega}(r) + \alpha(r) \end{bmatrix} v^{(m,n)}(r,t) + \bar{\eta}(r)u^{(m,n)}(r,t) + \frac{in}{r}\phi^{(m,n)}(r,t) = 0,$$
  
$$\begin{bmatrix} \frac{\partial}{\partial t} + in\bar{\Omega}(r) + \alpha(r) \end{bmatrix} \phi^{(m,n)}(r,t) + \frac{f^2l_{\rm R}^2}{r}\frac{\partial}{\partial r}ru^{(m,n)}(r,t) + \frac{inf^2l_{\rm R}^2}{r}v^{(m,n)}(r,t) = 0.$$
  
(C.1)

Here, the superscript (m, n) denotes the Fourier component of the field variable, expanded as in (20). The Fourier components are marched forward in time using a fourth-order Runge–Kutta scheme. The radial derivatives are approximated by second-order finite differences, on a grid with 4000 equally spaced elements. The outer radial boundary  $r_b$  is at  $20r_v/3$ ,  $20\bar{r}$  and  $10r_v$  for the RWS, Gaussian and Olivia simulations, respectively. At the outer radial boundary and the origin, we impose  $\phi^{(m,n)} = 0$ .

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The function  $\alpha(r)$  in (C.1) represents a sponge ring. Its specific form is given below:

$$\alpha(r) = \begin{cases} pf \sin^2 \left[ \frac{l\pi}{r_b} \left( r - qr_b \right) \right] & r > qr_b, \\ 0 & r < qr_b. \end{cases}$$
(C.2)

For the RWS and Gaussian simulations, p = 18.04, q = 7/8 and l = 4. For the Olivia simulation, p = 238.43, q = 0.9 and l = 5. The sponge ring damps the field variables near the outer radial boundary, thereby inhibiting the reflection of outward propagating inertia-buoyancy waves. The simulation results did not change for numerous test runs in which p was increased by a factor of ten, the boundary radius was doubled, or the radial grid-point spacing was halved.

The field variables are initialized so that the potential vorticity perturbation  $\Pi$  is given by (42). However, the initial value of  $\Pi$  does not completely specify the initial values of u, v and  $\phi$ . Of the infinite possibilities, we select that for which (u, v) has zero divergence  $\sigma$ , and for which  $\sigma$  has zero rate of change. That is, at t = 0, we impose

$$\sigma \equiv \frac{1}{r} \frac{\partial(ru)}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \lambda} = 0, \qquad \frac{\partial \sigma}{\partial t} = 0.$$
(C.3)

Since  $\sigma$  is initially zero, the initial velocity perturbation may be represented by a stream-function  $\psi$ ; i.e.

$$(u, v) = \left(-\frac{1}{r}\frac{\partial\psi}{\partial\lambda}, \frac{\partial\psi}{\partial r}\right).$$
(C.4)

From (11) and (C.4),  $\psi$  is related to the initial values of  $\Pi$  and  $\phi$  by the following:

$$N^{2}\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^{2}}\frac{\partial^{2}}{\partial\lambda^{2}}\right]\psi + \bar{\eta}\frac{\partial^{2}\phi}{\partial z^{2}} = \Pi.$$
(C.5)

The condition  $\partial \sigma / \partial t = 0$  may be written as an additional relation between  $\psi$  and  $\phi$ :

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\lambda^2}\right]\phi - \frac{1}{r}\frac{\partial}{\partial r}r\bar{\xi}\frac{\partial\psi}{\partial r} - \left[\frac{2\bar{\eta} - \bar{\xi}}{r^2}\right]\frac{\partial^2\psi}{\partial\lambda^2} = 0.$$
 (C.6)

Eq. (C.6) is derived with the aid of (7), (8) and (C.4).

We may transform Eqs. (C.5) and (C.6) into Fourier space, with the following results:

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right]\psi^{(m,n)} - \frac{\bar{\eta}}{f^2 l_{\rm R}^2}\phi^{(m,n)} = N^{-2}\Pi^{(m,n)},\tag{C.7}$$

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right]\phi^{(m,n)} - \frac{1}{r}\frac{\partial}{\partial r}r\bar{\xi}\frac{\partial}{\partial r}\psi^{(m,n)} + n^2\left[\frac{2\bar{\eta} - \bar{\xi}}{r^2}\right]\psi^{(m,n)} = 0.$$
(C.8)

Given  $\Pi^{(m,n)}$ , and the boundary conditions  $\phi^{(m,n)}|_{0,r_b} = \psi^{(m,n)}|_{0,r_b} = 0$ , we solve (C.7) and (C.8) simultaneously for  $\phi^{(m,n)}$  and  $\psi^{(m,n)}$ . The initial conditions for  $u^{(m,n)}$  and  $v^{(m,n)}$  are then obtained from the Fourier transform of (C.4):

$$u^{(m,n)} = -\frac{in}{r}\psi^{(m,n)}, \qquad v^{(m,n)} = \frac{\partial}{\partial r}\psi^{(m,n)}.$$
(C.9)

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