Notes and Correspondence
Hurricane intensity in the Ooyama (1969) paradigm

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This note derives an analytical approximation for steady-state hurricane intensity in the three-layer model of Ooyama (1969). As in the more realistic but involved theory of Emanuel (1986), the square of the maximum wind speed is roughly proportional to the ratio of entropy to momentum exchange coefficients, times a measure of the ambient thermal disequilibrium between the sea surface and the upper troposphere. The analytical approximation compares favourably to a set of three-layer numerical simulations that covers a broad range of parameter space. Limitations of the analysis are briefly addressed, and a supergradient wind correction is estimated. Copyright © 2010 Royal Meteorological Society

Key Words: tropical cyclones; steady-state maintenance; simple models

Received 19 January 2010; Revised 16 June 2010; Accepted 16 June 2010; Published online in Wiley Online Library 16 August 2010

Citation: Schecter DA. 2010. Hurricane intensity in the Ooyama (1969) paradigm. *Q. J. R. Meteorol. Soc.* **136**: 1920–1926. DOI:10.1002/qj.657

1. Introduction

The simple tropical cyclone model of Ooyama (1969–O69 hereafter) provided early insight into the response of hurricanes to variations of sea-surface temperature and surface-exchange coefficients. Numerical results from the model (O69; DeMaria and Pickle, 1988; Schecter and Dunkerton, 2009–SD09 hereafter) are qualitatively consistent with intensity predictions given by the steady-state theory of Emanuel (Emanuel, 1986–E86 hereafter; Rotunno and Emanuel, 1987; Emanuel, 1988; Emanuel, 1995–E95 hereafter). The similarities between O69 and E86 are comforting, but are not fully explained in the literature. The O69 model has three layers, whereas the E86 model is continuous. The O69 model involves a crude cumulus parametrization, whereas the E86 model directly accounts for cloud moisture.

This note is an effort to explain the scaling of hurricane intensity in the context of O69, with minimal complications. An analytical theory is followed by a successful comparison to numerical results. The analysis has virtue in its simplicity, but does not incorporate several features of realistic hurricanes that are known to affect intensity. Appendix A1 briefly addresses the neglected presence of supergradient wind in the boundary layer, but deviation from axisymmetric flow, mid-level ventilation, and other subtleties are beyond the modest scope of this note. Those interested may readily consult a growing body of literature on the aforementioned issues (Bister and Emanuel, 1997; Persing and Montgomery, 2003; Smith *et al.*, 2008; Smith and Montgomery, 2008; Bryan and Rotunno, 2009a,b; Tang and Emanuel, 2010; Montgomery *et al.*, 2010).

2. Estimate of hurricane intensity

Figure 1 illustrates an axisymmetric steady-state hurricane in the O69 paradigm. The model consists of two ‘shallow-water’ layers over a flat boundary layer. The mass density of the upper layer is a fraction ϵ of the common mass density ρ of the boundary and middle layers. An ocean with constant surface temperature sits beneath the vortex. In the boundary layer, air spirals radially inward while experiencing drag and gaining moisture (moist entropy) from the underlying ocean. The inflow converges into a cumulus updraught, accentuated near the radius of maximum wind. The cumulus air mass rarefies and ascends into the upper layer, where it proceeds to flow radially outward.

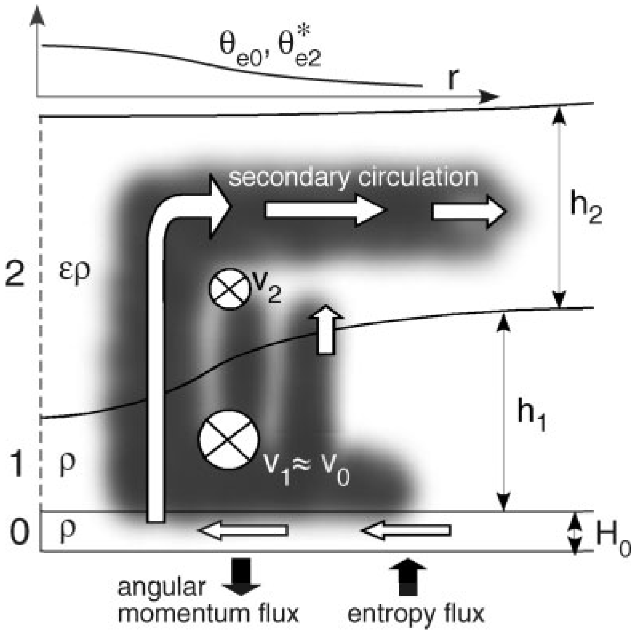

Figure 1. Illustration of a steady-state hurricane in the three-layer model of Ooyama (1969). The boundary, middle and upper layers are labelled 0, 1 and 2, respectively. The primary azimuthal circulation (v_m) is into the page. White arrows show the secondary circulation. The grey clouds crudely depict moist convection in the hurricane, which the model represents by the rarefaction of lower tropospheric air, and its ascent into the upper layer.

Table I. Correspondence between equations of this note and O69.

Description	This note	Ooyama (1969)
Deep convective neutrality	Eq. (1)	Sections 8, 10 Fig. 6 (194 hr)
Gradient (cyclostrophic) balance	Eqs (2)	Eq. (3.1), Eq. (3.2)
Geopotential perturbations	Eqs (3)	Eqs (2.9), Eq. (2.10)
Angular momentum equation in the boundary layer	Eq. (4)	Eq. (3.19)
Moist entropy equation in the boundary layer	Eq. (5)	Eq. (7.4)
Saturation equivalent potential temperatures	Eqs (6)	Eq. (7.5), Eq. (6.4)
Mass continuity equation in the upper layer	Eq. (10)	Eq. (2.6)
Angular momentum equation in the upper layer	Eq. (11)	Eq. (3.5)

The following derivation of steady-state hurricane intensity is reasonably self-contained, but some familiarity with O69 would likely benefit the reader. For convenient reference, Table I indicates where O69 explicitly or implicitly presents the fundamental vortex equations used in the analysis. The O69 equations include lateral eddy-fluxes and shearing stresses at the top and bottom of the middle layer. In principle, both items can affect hurricane intensity, but are neglected in this note.

For simplicity, let us assume that the thermal structure of the steady state approximately satisfies the following condition of deep convective neutrality:

$$\theta_{e0}(r) = \theta^*_{e2}(r), \tag{1}$$

in which θ_{e0} is the equivalent potential temperature of the boundary layer, θ^*_{e2} is the saturation equivalent potential temperature of the upper layer, and r is the radius from the centre of the vortex. Condition (1) is analogous but not equivalent to that of slantwise convective neutrality in Emanuel's more realistic model of steady-state hurricanes (E86). In the O69 cumulus parametrization, deep convective neutrality eliminates the entrainment of mid-level air by cumulus updraughts, and thereby eliminates convergence in the middle layer. In principle, intensification or spin-down would occur if θ_{e0} were greater or less than θ^*_{e2} (O69; Smith, 2000). Moreover, condition (1) appears to hold fairly well at late times in the original numerical simulations of Ooyama (Figure 6 of O69). Ultimately, we apply condition (1) only to the eyewall region of the hurricane, which contains the radius of maximum wind.

Let us further suppose that the ratio of planetary to relative vorticity is small, and that all three layers of the hurricane satisfy cyclostrophic balance in the central region of interest. In the O69 model, cyclostrophic balance translates into

$$\frac{\mathrm{d}\phi_1}{\mathrm{d}r} = \frac{v_1^2}{r} = \frac{v_0^2}{r} \quad \text{and} \quad \frac{\mathrm{d}\phi_2}{\mathrm{d}r} = \frac{v_2^2}{r}, \tag{2}$$

in which v_m and ϕ_m are the azimuthal velocity and geopotential perturbations of layer m. The geopotential perturbations are defined by

$$\left.\begin{aligned} \phi_0 = \phi_1 &\equiv g(h_1 - H_1) + \epsilon g(h_2 - H_2) \\ \text{and} \\ \phi_2 &\equiv g(h_1 - H_1) + g(h_2 - H_2), \end{aligned}\right\} \tag{3}$$

in which h_m and H_m are the local and ambient thicknesses of layer m, and g is the gravitational acceleration.

The legitimacy of assuming cyclostrophic balance in the hurricane core generally requires dominance of centrifugal acceleration over radial advection and other terms (excluding the geopotential gradient) in the radial momentum equation. This condition is known to be questionable in the boundary layer of an arbitrary hurricane (e.g. Smith and Montgomery, 2008). The constraint $v_0 = v_1$ implied by Eq. (2), which follows from extending cyclostrophic balance (or gradient balance) to the boundary layer, is therefore open to criticism. Appendix A1 examines the error associated with this approximation. For now, let it suffice to view $v_0 = v_1$ as a provisional characteristic of the steady state, consistent with O69.

Continuing along, if the azimuthal wind speed is much greater than the radial wind speed in the boundary layer, then the steady-state angular momentum equation in the boundary layer reduces to

$$u_0 = \frac{-C_D v_0^2}{H_0 \zeta_0}, \tag{4}$$

in which u_m and $\zeta_m \equiv r^{-1}\mathrm{d}(rv_m)/\mathrm{d}r$ are the radial velocity and relative vorticity of layer m. The dimensionless parameter C_D in Eq. (4) is the momentum exchange coefficient for air–sea interaction. As before, planetary vorticity has been neglected under the assumption of high relative vorticity inside the central region of the hurricane.

Let us also assume that any downward mass flux of low-entropy air into the boundary layer is negligible inside a loosely defined perimeter of the vortex core. Then, the steady-state equation for equivalent potential temperature in the core region of the boundary layer reduces to

$$u_0 \frac{\mathrm{d}\theta_{e0}}{\mathrm{d}r} = C_E v_0 \frac{\theta^*_{es} - \theta_{e0}}{H_0}, \tag{5}$$

in which θ^*_{es} is the saturation equivalent potential temperature at the sea surface, and C_E is the dimensionless surface-exchange coefficient for moist entropy. Equation (5) notably implies that (in steady state) an unsaturated boundary layer is required for the coexistence of radial inflow and a non-zero entropy gradient (i.e. non-zero $\mathrm{d}\theta_{e0}/\mathrm{d}r$).

In the O69 model, the saturation equivalent potential temperatures at the sea surface and in the upper layer are related to the geopotential perturbations by

$$\left.\begin{aligned} \theta^*_{es} &= \overline{\theta}^*_{es} - \frac{\beta}{c_p}\phi_1, \\ \theta^*_{e2} &= \overline{\theta}^*_{e2} + \frac{\alpha}{c_p}(\phi_2 - \phi_1), \end{aligned}\right\} \tag{6}$$

in which α and β are positive dimensionless constants, c_p is the isobaric specific heat of air, and overbars denote ambient values. The top equation expresses the elevation of saturation equivalent potential temperature at lower values of the sea-surface pressure p_s, since $\rho\phi_1 = p_s - \overline{p}_s$. The bottom relation expresses the association of warming with expansion of the rarefied upper layer, since $\phi_2 - \phi_1 = g(1-\epsilon)(h_2 - H_2)$.

Substituting Eqs (1), (4) and (6) into Eq. (5), and appealing to cyclostrophic balance (2) yields

$$v_0^3 \frac{1 - v_2^2/v_0^2}{\mathrm{d}(rv_0)/\mathrm{d}r} = \frac{c_p}{\alpha}\frac{C_E}{C_D}\left[\overline{\theta}^*_{es} - \overline{\theta}^*_{e2} - \frac{\alpha}{c_p}(\phi_2 - \phi_1) - \frac{\beta}{c_p}\phi_1\right]. \tag{7}$$

Let V_m denote the characteristic azimuthal velocity of the vortex in layer m. By Eq. (2), we estimate that $\phi_m \sim -V_m^2$. By Eq. (7), we further estimate that

$$V_0^2 \sim \frac{C_E}{C_D}\frac{c_p}{\alpha S}\frac{\overline{\theta}^*_{es} - \overline{\theta}^*_{e2}}{1 + \frac{C_E}{C_D}\left(1 - \frac{\beta}{\alpha S}\right)}, \tag{8}$$

in which $S \equiv 1 - V_2^2/V_0^2$. The quantitative accuracy of the above relation depends in part on the 'empirical' validity of substituting unity for several unknown scaling coefficients in the derivation. Closure requires a formula for S, which measures the vertical shear of the primary circulation. Two possibilities are considered in section 4 of this note. In both cases, and in the numerical simulations of section 3, S varies little with V_0.

Equation (8) implies that V_0^2 is proportional to C_E/C_D, as the ratio tends toward zero, times a measure of the ambient thermal disequilibrium between the sea surface and the upper troposphere. This result is similar to Eq. (43) of E86 (or Eq. (16) of E95), as explained in Appendix A2.* It is worth noting that $\overline{\theta}^*_{es} - \overline{\theta}^*_{e2}$ is not directly related to the ambient convective available potential energy (CAPE), which is unspecified in the preceding analysis, and may be zero provided that condition (1) extends to the environment.

Clearly, we have not derived a rigorous analytical solution for the steady state. Equation (8) is best viewed as a reasonable hypothesis, founded on basic physical considerations. The merit of this formula must be judged by comparison to numerical simulations.

3. Comparison to numerical simulations

Figure 2 compares the theoretical vortex intensity (Eq. (8)) to the tropical cyclone simulations of SD09. The SD09 simulations are based on a three-layer model with an O69-like cumulus parameterization. The layer parameters are $H_0 = 1$ km, $H_1 = H_2 = 5$ km, and $\epsilon = 0.9$. The computational domain is a 2000×2000 km periodic box, with 3.9 km horizontal grid increments. The surface-exchange coefficients vary with the boundary-layer wind speed $|\mathbf{v_0}|$ according to an O69-like formula,

$$C_{D,E} = C^*_{D,E}(1 + |\mathbf{v_0}|/v_*), \tag{9}$$

in which C^*_D and C^*_E are adjustable parameters, whereas v_* is always $8.33\,\mathrm{m\,s^{-1}}$. (Interested readers may consult Black *et al.*, 2007, for a more realistic description of air–sea exchange.) The simulations begin with turbulent initial conditions and generate tropical cyclones over time-scales of days to months. The parameter regime covers over one decade of C_E/C_D, numerous values of $\overline{\theta}^*_{es}$ that span a broad range of sea-surface temperatures, and several tropical values of the Coriolis parameter f. A distinct parameter set is represented by a distinct symbol on the scatter plot. Different data points for the same parameters correspond to different initial conditions.

The plotted simulation values of vortex intensity are given by $V_{sim} \equiv \langle v_0 \rangle_t$, in which v_m is the azimuthal mean of the azimuthal velocity of layer m. The polar coordinate system which defines the azimuthal velocity is centred at the instantaneous point of minimum surface pressure. The operator $\langle \ldots \rangle_t$ takes the time average of the enclosed variable, evaluated at the radius of maximum v_0 (henceforth r_{mw}), after the vortex reaches a statistically stationary state. The theoretical vortex intensity (V_{th}) on the horizontal axis is the steady-state time average of the right-hand side of Eq. (8), with the shear parameter $S \equiv 1 - v_2^2/v_0^2$ evaluated at r_{mw}, and $\overline{\theta}^*_{es} - \overline{\theta}^*_{e2}$ set equal to the domain average of $\theta^*_{es} - \theta^*_{e2}$.† The values of α and β are 10 and 2, respectively, and $c_p = 1005.7\,\mathrm{J\,kg^{-1}K^{-1}}$. Evidently, Eq. (8) provides an accurate description of the simulation results in SD09.

We may also compare Eq. (8) to a detailed simulation result in O69 (far-right column of Figure 6). The O69 hurricane under consideration has the following properties at the 60 km radius of maximum wind: $v_0 \simeq 49\,\mathrm{m\,s^{-1}}$ and

*Recent cloud-resolving numerical simulations suggest that hurricane intensity may be less sensitive to C_E/C_D than suggested by E86 and the present analysis, due to lateral eddy-fluxes, non-axisymmetric flow, or relatively involved boundary layer dynamics (Bryan and Rotunno, 2009a; Montgomery *et al.*, 2010).

†Be aware of the notational discrepancy with SD09, in which an overbar strictly denotes the *initial value* of the domain average.

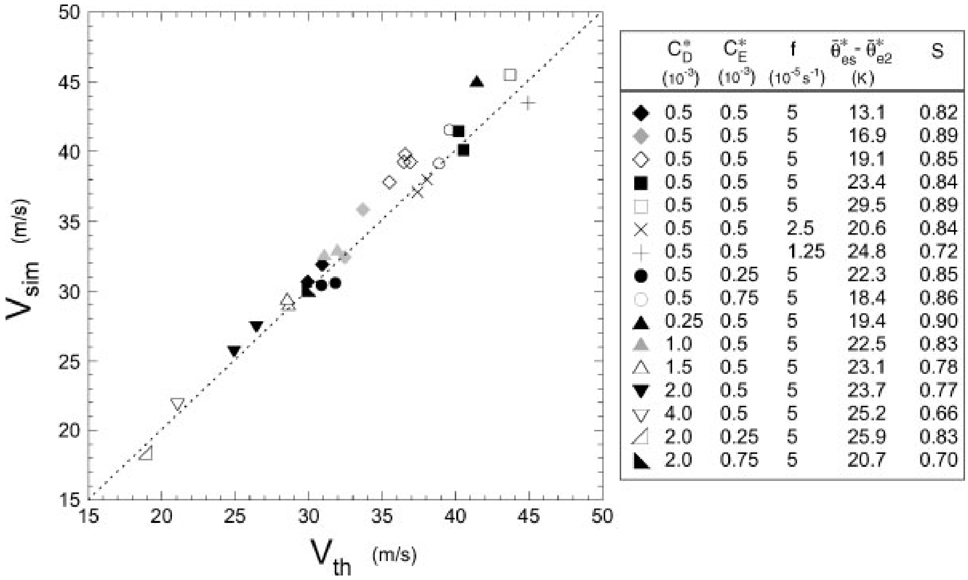

Figure 2. The maximum tangential wind speeds of the simulated tropical cyclones of SD09 (V_{sim}) versus the theoretical wind speeds (V_{th}) of Eq. (8). Each symbol corresponds to a unique set of values for C_D^*, C_E^*, f and $\overline{\theta}^*_{\mathrm{es}}$. The equilibrium values of $\overline{\theta}^*_{\mathrm{e2}}$ and S vary slightly with initial conditions; the tabulated values of $\overline{\theta}^*_{\mathrm{es}} - \overline{\theta}^*_{\mathrm{e2}}$ and S are averages over a given set. The dotted line represents the curve $V_{\mathrm{sim}} = V_{\mathrm{th}}$. In general, the simulations have 3.9-km horizontal grid increments; simulation + has 2-km increments.

$S \simeq 0.5$. The ratio of surface-exchange coefficients is unity, and $\overline{\theta}^*_{\mathrm{es}} - \overline{\theta}^*_{\mathrm{e2}}$ measured at $r = 400$ km is about 20 K. With α, β and c_p given as before, Eq. (8) predicts an agreeable wind speed of 50 m s^{-1}.

The apparent accuracy of Eq. (8) is remarkable, given that the estimate was built upon simplified conditions for the steady state, and set various unknown coefficients of proportionality to one. In fairness, successful comparison to SD09, which uses a model less constrained than the O69 prototype, involves some cancellation of error. To begin with, SD09 hurricanes have persistent asymmetrical fluctuations in their statistical equilibria, which violates the fundamental premise of axisymmetry. Although condition (1) holds well (on average) in the eyewall, which contains the radius of maximum wind, the condition $\mathrm{d}\theta_{\mathrm{e0}}/\mathrm{d}r = \mathrm{d}\theta^*_{\mathrm{e2}}/\mathrm{d}r$ is fragile and generally less accurate. Furthermore, the boundary layer develops a non-trivial degree of supergradient flow ($v_0 > v_1$) for those cases with exceptionally large values of C_D^*. Hence, the preceding derivation of Eq. (8) provides substantial insight into steady-state maintenance, but falls short of telling the complete story.

4. The S-closure

As mentioned earlier, the scaling theory is incomplete without a formula for the shear parameter S. In order to obtain some degree of closure, let us consider the secondary circulation of the hurricane core in greater detail. The cumulus mass flux out of the boundary layer is proportional to the convergence of u_0. The upper layer is steady only if the divergence of its (outward) radial mass flux cancels the influx of cumulus air mass from below. For a neutral axisymmetric vortex in the O69 model, this condition is tantamount to

$$\epsilon \frac{\mathrm{d}(r u_2 h_2)}{\mathrm{d}r} = -\frac{\mathrm{d}(r u_0 H_0)}{\mathrm{d}r},$$

which upon integrating outward from $r = 0$ yields

$$u_2 = -\frac{u_0 H_0}{\epsilon h_2}. \tag{10}$$

Furthermore, the divergence of the radial flux of angular momentum in the upper layer must cancel the cumulus influx from below. Neglecting planetary vorticity, this condition may be written as:

$$\epsilon \frac{\mathrm{d}(r^2 u_2 v_2 h_2)}{\mathrm{d}r} = -\frac{\mathrm{d}(r u_0 H_0)}{\mathrm{d}r} r v_0. \tag{11}$$

Substituting Eqs (10) and (4) into (11) and integrating outward from $r = 0$ yields

$$\frac{v_2}{v_0} = 1 - \frac{1}{r^3 C_D v_0^3}\frac{\mathrm{d}(r v_0)}{\mathrm{d}r}\int_0^r \tilde{r}^2 C_D v_0^2\,\mathrm{d}\tilde{r}. \tag{12}$$

Suppose that the azimuthal velocity in the boundary layer is Rankine ($v_0 \propto r$) within the radius of maximum wind r_{mw}, and C_D satisfies Eq. (9). Then, the right-hand side of Eq. (12) may be solved at r_{mw}^- to find a shear parameter S that varies from 0.56 to 0.64 as the maximum of v_0 decreases from infinity to zero. This range of values is somewhat below that found for S in the numerical simulations of SD09, suggesting that the shear parameter may be sensitive to fine details of v_0, or to asymmetrical fluctuations which exist in the statistical equilibrium.

An alternative closure begins by considering a variant of the thermal wind relation in the vicinity of r_{mw}, where cyclostrophic balance and condition (1) are assumed valid:

$$1 - \frac{v_2^2}{v_0^2} = \frac{\beta}{\alpha}\frac{\mathrm{d}\theta^*_{\mathrm{e2}}}{\mathrm{d}\theta^*_{\mathrm{es}}} = \frac{\beta}{\alpha}\frac{\mathrm{d}\theta_{\mathrm{e0}}}{\mathrm{d}\theta^*_{\mathrm{es}}}. \tag{13}$$

Suppose that θ_{e0} decays with increasing radius at and beyond r_{mw}, specifically as though temperature and relative humidity were constant (cf. E86). This would imply that θ_{e0}

depends on pressure alone in the outer region of the vortex. Linearizing the pressure dependence yields

$$\theta_{e0} = \overline{\theta}_{e0} - \frac{\beta'}{c_p}\phi_1 , \qquad r \geq r_{mw} , \tag{14}$$

in which β' is a constant determined by the ambient conditions. Substituting Eq. (14) into the right-hand side of Eq. (13) gives $S = \beta'/\alpha$. Since plausible values of β' are comparable to β, the closure under consideration yields substantially smaller values for S (around 0.2) than are observed at r_{mw} in the simulations. An implicit outward radial decay of relative humidity (or temperature) in the boundary layer would produce a larger value of $d\theta_{e0}/d\theta^*_{es}$, and a larger value of S by Eq. (13). The value of S would also increase if $d\theta^*_{e2}/d\theta_{e0} > 1$, in violation of condition (1).

5. Conclusion

We have shown that a 'back-of-the-envelope' analysis provides substantial insight into the scaling of hurricane intensity in the O69 paradigm. As in E86, our derived expression for the square of the maximum wind speed of a steady-state hurricane (Eq. (8)) is roughly proportional to the ratio of entropy to momentum exchange coefficients, times a measure of the ambient thermal disequilibrium between the sea surface and the upper troposphere. The wind speed formula compared favourably to a large and diverse set of three-layer numerical simulations. Because of its simplicity, the scaling theory presented here may have some pedagogical value. However, any simple theory of an atmospheric system has limited applicability. Appendix A1 addresses one important limitation of Eq. (8) caused by neglecting supergradient flow in the boundary layer. Other deficiencies shared with E86 were noted in section 1, along with pertinent references.

Acknowledgements

The author thanks Prof. R. K. Smith and Dr. J. Persing for their constructive critiques of this note during the review stage of publication. The author also thanks Prof. W. H. Schubert and Dr. T. J. Dunkerton for their helpful comments on an earlier draft. This work was supported by NSF grant ATM-0750660.

Appendices

A1. Supergradient flow

It is well known that supergradient flow can occur in the boundary layer of a realistic hurricane (Smith *et al.*, 2008; Smith and Montgomery, 2008; Bryan and Rotunno, 2009b). By contrast, the original O69 model (used in section 2) imposes gradient balance on the primary circulation of the entire vortex. In this appendix, we remove the balance constraint on v_0 and estimate the degree of supergradient flow near the radius of maximum wind r_{mw}, given C_D and a few geometrical parameters of the vortex.

Including terms neglected in O69, the steady-state radial momentum equation for the boundary layer has the following form (cf. SD09):

$$\begin{aligned} \frac{v_0^2}{r}\left(1+\frac{fr}{v_0}\right) - \frac{d}{dr}\left(\frac{u_0^2}{2}\right) - \frac{C_D\,|\mathbf{v}_0|\,u_0}{H_0} - \frac{u_0 w_-}{H_0} \\ = \frac{d\phi_1}{dr} = \frac{v_1^2}{r}\left(1+\frac{fr}{v_1}\right). \end{aligned} \tag{A1}$$

The new variable w_- (times ρ) is the downward mass flux on top of the boundary layer; the value of w_- is non-zero (and positive) only in regions of subsidence. The term $-u_0w_-/H_0$, which may be unfamiliar to some readers, represents radial drag due to momentum mixing with the middle layer, under the assumption that $u_1 \ll u_0$. The far right-hand side of Eq. (A1) assumes that gradient balance remains a valid approximation in the middle layer.

By definition, supergradient flow is the condition where the centrifugal (plus Coriolis) force exceeds the inward force imposed by the radial pressure gradient. By Eq. (A1), it exists in the boundary layer *iff*

$$\frac{d}{dr}\left(\frac{u_0^2}{2}\right) > -\frac{C_D|\mathbf{v}_0|u_0 + u_0w_-}{H_0},$$

that is, *iff* the Lagrangian deceleration of radial inflow exceeds the deceleration due to surface friction and momentum mixing with the middle layer. For the model under consideration, supergradient flow in the boundary layer is tantamount to the condition $v_0 > v_1$.

To estimate the degree to which v_0 might exceed v_1 near r_{mw}, it is helpful to reformulate Eq. (A1). The first step is to consider the following steady-state angular momentum equation in the boundary layer:

$$u_0 = -\frac{C_D\,|\mathbf{v}_0|\,v_0}{H_0\zeta_0(1+f/\zeta_0)} - \frac{(v_0-v_1)w_-}{H_0\zeta_0(1+f/\zeta_0)}. \tag{A2}$$

The second step is to note that the vertical mass flux is positive ($w_- = 0$) in the vicinity of r_{mw}. Substituting Eq. (A2) and $w_- = 0$ into Eq. (A1) yields

$$\begin{aligned} \frac{v_1^2}{v_0^2} = \frac{1+fr/v_0}{1+fr/v_1} - \frac{r}{v_0^2\,(1+fr/v_1)} \\ \times\left[\frac{1}{2}\frac{d}{dr}\left(\frac{C_D|\mathbf{v}_0|v_0}{H_0\zeta_0(1+f/\zeta_0)}\right)^2 - \left(\frac{C_D|\mathbf{v}_0|}{H_0}\right)^2\frac{v_0}{\zeta_0(1+f/\zeta_0)}\right]. \end{aligned} \tag{A3}$$

The first term in the square brackets, $d(u_0^2/2)/dr$, is typically positive near r_{mw}, whereas the second term, $C_D|\mathbf{v}_0|u_0/H_0$, is negative.

Suppose that the Rossby number (near r_{mw}) is much greater than unity, such that $v_m/fr \gg 1$ and $\zeta_m/f \gg 1$. Further, suppose that $|u_0| \lesssim |v_0|$. We estimate that $\zeta_0 \sim v_0/r_{mw}$ and $d\zeta_0/dr \sim -v_0/r^2_{mw}\delta$, in which $r_{mw}\delta$ is the radial decay length of ζ_0 near r_{mw}. If $\delta \ll 1$, then the first term of Eq. (A3) in square brackets likely dominates the second. Taking this and our previous assumptions into consideration, we obtain

$$\frac{v_1^2}{v_0^2} - 1 \sim -\frac{1}{\delta}\left(\frac{C_D r_{mw}}{H_0}\right)^2. \tag{A4}$$

The above estimate implies some degree of supergradient flow ($v_0 > v_1$) at r_{mw}. A severe violation of gradient balance seems possible, but a plausible set of parameters such as $\{r_{mw}/H_0, C_D, \delta\} = \{40, 0.002, 0.1\}$ gives only a modest error of $v_1^2/v_0^2 - 1 \sim -0.064$. According to Eq. (A4), increasing surface friction has the potential to enhance supergradient flow; however, an attendant variation of r_{mw} or δ could modify the effect.

Figure 2 suggests that the maximum wind speed of a tropical cyclone does not diverge from the balanced flow estimate (Eq. (8)) as C_D^* becomes exceptionally large in the SD09 model, which permits supergradient wind. This result is interesting, but is partially due to numerical error at relatively high values of C_D^*, where r_{mw} is about 10 km (SD09). Repeating a subset of the simulations after reducing the grid spacing from 3.9 to 2 km produces values of V_{sim}/V_{th} (defined in section 3) that clearly increase with C_D^*.[‡] Specifically, as C_D^* grows from 0.5×10^{-3} to 4×10^{-3}, the value of V_{sim}/V_{th} grows from 1.1 to 1.2.

A2. The E86 wind speed formula

In his seminal paper, Emanuel derived the following formula for the maximum tangential wind speed of a hurricane (Eq. (43) of E86; Eq. (16) of E95):

$$V_{max}^2 = \frac{C_E}{C_D}\gamma L_{lv} q_{as}^*(1 - \mathrm{RH}_{as}) \times \frac{1 - \frac{1}{4}\frac{f^2 r_o^2}{\xi R T_B}}{1 - \frac{1}{2}\frac{C_E}{C_D}\gamma\frac{L_{lv}q_{as}^*(1-\mathrm{RH}_{as})}{\xi R T_s}}. \quad \text{(A5)}$$

As usual, C_E and C_D are the surface-exchange coefficients for entropy and momentum, respectively. L_{lv} is the latent heat of vaporization, R is the gas constant of air, and f is the Coriolis parameter. T_B and T_s are the absolute temperatures along the tops of the subcloud layer and the surface layer, respectively. The variables q_{as}^* and RH_{as} are ambient values of the saturation mixing ratio and relative humidity on top of the surface layer; the relative humidity is assumed constant along the top of the surface layer, at and beyond the radius of maximum wind. The variable r_o is the outer radius of the storm near sea-level. The thermodynamic efficiency γ and the ξ-parameter (Eqs (37) and (38) of E86) are defined by

$$\gamma \equiv \frac{T_B - T_o}{T_B} \quad \text{and} \quad \xi = 1 - \gamma\left(1 + \frac{L_{lv}q_{as}^*\mathrm{RH}_{as}}{RT_s}\right). \quad \text{(A6)}$$

Here, T_o is the average outflow temperature of convective air parcels, which is roughly the tropopause temperature. How does the E86 wind speed formula compare with our estimate based on O69?

The leading factor $(C_E/C_D)\gamma$ in Eq. (A5) is analogous to $(C_E/C_D)(\overline{\theta}_{es}^* - \overline{\theta}_{e2}^*)$ in Eq. (8). On the other hand, the E86 formula contains an explicit proportionality to $1 - \mathrm{RH}_{as}$, and a weak dependence on fr_o. The absence of the factor $1 - \mathrm{RH}_{as}$ in Eq. (8) is not a critical deficiency. Whereas the wind speed explicitly vanishes under saturated conditions in E86, it implicitly vanishes in Eq. (8) by way of the neutrality constraint[§]. The absence of the factor $1 - f^2 r_0^2/4\xi RT_B$ is also acceptable, since it is very close to unity for typical hurricane parameters. It is worth noting that the first S-closure of section 4 (Eq. (12)) circumvents direct consideration of relative humidity in the outer boundary layer. Furthermore, the assumption $\phi_m \sim -V_m^2$ obviates direct consideration of r_o and the outer Coriolis force, for the purpose of estimating the maximum wind speed.

Those interested in directly relating the derivation of Eq. (8) to the analysis of E86 may first consider the following rudimentary form of Eq. (7), which stems from Eqs (4) and (5) alone:

$$v_0^2 = -\frac{1}{2r^2}\frac{C_E}{C_D}\frac{dM_0^2}{d\theta_{e0}}(\theta_{es}^* - \theta_{e0}). \quad \text{(A7)}$$

Here, M_0 is the absolute angular momentum of the boundary layer, which reduces to rv_0 in the core region of interest. In the framework of E86, a similar formula applies along the top of the subcloud layer (Eq. (13) of E95). However, the E86 formula incorporates a variant of the thermal wind relation that equates the analogue of

$$-\frac{dM_0^2}{d\theta_{e0}}\frac{\theta_{e0}}{2c_p r^2}$$

to the parameter $T_B - T_o$ (Eq. (13) of E86; Eq. (5) of E95). A thermal wind relation of such form requires a condition of slantwise convective neutrality of saturated air above the subcloud layer, and is crucial to the conversion of Eq. (A7) into Eq. (A5). In contrast, this note implicitly uses the formula

$$-\frac{1}{2r^2}\frac{dM_0^2}{d\theta_{e0}} \sim \frac{c_p}{\alpha S},$$

and offers two possible solutions for S.

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[‡]For all simulations in the subset, $C_E^* = 0.5 \times 10^{-3}$, $f = 5 \times 10^{-5}\,\mathrm{s}^{-1}$, and $\overline{\theta}_{es}^* = 372$ K. The values of $\overline{\theta}_{e2}^*$ are initially equal, but change over time, as mentioned previously.

[§]Taking the neutrality constraint (Eq. (1)) into consideration, the saturation condition $\theta_{es}^* - \theta_{e0} = 0$ amounts to $\theta_{es}^* - \theta_{e2}^* = 0$ in the region of interest. Using Eqs (6) and the assumption $\phi_m \sim -V_m^2$, we may rewrite the saturation condition as follows: $\overline{\theta}_{es}^* - \overline{\theta}_{e2}^* = V_0^2(\alpha S - \beta)/c_p$. Substituting this equation into (8), one finds that $V_0 = 0$ is the only solution.

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