A Brief Critique of a Theory Used to Interpret the Infrasound of Tornadic Thunderstorms

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Abstract

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It has been proposed that the 0.5-10 Hz infrasound emitted by a severe storm is primarily generated by the *axisymmetric* oscillations of a tornado. This interpretation is challenged by a critical review of its theoretical foundation. A basic linear analysis shows that the principal axisymmetric oscillations of a subsonic, columnar vortex (axisymmetric Kelvin modes) can not excite acoustic radiation. Numerical experiments further show that axisymmetric radiation is shaped primarily by the impulse that triggers the emission, not the properties of the vortex.

1. Introduction

1.1 Background

Decades ago, Abdullah presented a seemingly straightforward theory for the "musical sound" of a tornado [Abdullah 1966 (A66)]. Modeling the tornado as a compressible Rankine vortex, he derived a simple frequency formula for a fast branch of core vibrations that generate sustained acoustic emissions. For the particular case of axisymmetric vibrations with a large ratio of vertical-to-radial wavelengths, the A66 frequency formula reduces to

$$f_n = \frac{(4n+5)c_o}{8a},\tag{1}$$

in which n is a non-negative integer that increases with the characteristic radial wavenumber, c_o is the ambient sound speed, and a is the radius of the vortex core. The derivation of Eq. (1) reasonably assumes that the squared Mach number of the vortex is much less than unity.

Substituting the typical values $c_o = 330 \text{ m s}^{-1}$ and a = 100 m into Eq. (1) yields $f_0 = 2.1 \text{ Hz}$. This value of f_0 falls in the 0.5-10 Hz frequency band where tornadic thunderstorms are said to emit distinct infrasound [Bedard 2005; Bedard et al. 2004]. It is therefore tempting to conclude that the infrasound comes from axisymmetric vortex oscillations [ibid].

1.2 Objection and Objectives

Equation (1) predicts that the peak frequencies of tornado infrasound (in the neighborhood of 1 Hz) are inversely proportional to a, but have no dependence on windspeed. Both predictions are reportedly consistent with observations of infrasound emitted from the vicinity of a tornado.¹ This does not mean that A66 provides the correct explanation. Appealing to

¹Supporting evidence for inverse proportionality between frequency and a can be found in Bedard [2005] and Bedard et al. [2004]. Insensitivity of frequency (and amplitude) to windspeed was reported to the author by an anonymous referee for *Monthly Weather Review*.

basic intuition, it seems odd that the frequency spectrum for any class of modes attributable to the vortex does not change (provided a is fixed) as the windspeed goes to zero.

One purpose of this paper is to point out a subtle misstep in the derivation of Eq. (1) that casts doubt upon its fundamental credibility. In addition, this paper will show that the principal axisymmetric oscillations of a subsonic Rankine vortex (axisymmetric Kelvin modes) do not emit acoustic radiation. Finally, this paper will present numerical evidence that the axisymmetric radiation emitted after a generic disturbance of a Rankine vortex carries no significant vortex signature.

1.3 Notational Conventions

Insofar as possible, our notational conventions abide by contemporary norms. The variables r, φ and z denote radius, azimuth and height in a cylindrical coordinate system centered on the vortex [see Fig. 1]. The variable t denotes time. Overbars and primes are used to represent the basic state and perturbation of a generic fluid variable G, such that $G \equiv \overline{G} + G'$. Specific fluid variables include the radial velocity u, the azimuthal velocity v, the axial (vertical) velocity w, the pressure p, and the potential temperature θ .²

2. Reexamination of A66

2.1 Matching Conditions at the Boundary of the Vortex Core

Following the derivation of Eq. (1) in A66, one arrives at a step where each component of the velocity perturbation is forced to be continuous at the core radius of the Rankine tornado [Eq.(31) of A66]. This continuity constraint may seem reasonable, but imposing it on the azimuthal velocity perturbation (v') is a mathematically subtle issue.

 $^{^2\}mathrm{In}$ contrast to our notation, A66 uses the letters u and v to represent azimuthal and radial velocities, respectively.

For simplicity, let us restrict our discussion to axisymmetric disturbances. Neglecting φ -derivatives, the prognostic equation for v' in A66 takes the following linearized form:

$$\frac{\partial v'}{\partial t} = -\bar{\zeta}u',\tag{2}$$

in which $\overline{\zeta}$ is the axial vorticity of the basic state. For Rankine vortices,

$$\bar{v}(r) \equiv \begin{cases} \Omega r & r < a, \\ \Omega a^2/r & r > a, \end{cases}$$
(3)

in which Ω is the angular rotation frequency of the core. It follows that

$$\bar{\zeta}(r) \equiv \frac{1}{r} \frac{d(r\bar{v})}{dr} = \begin{cases} 2\Omega & r < a, \\ 0 & r > a. \end{cases}$$
(4)

A66 requires consistency of Eq. (2) with the aforementioned continuity constraints on u' and v'. Because $\bar{\zeta}$ is discontinuous at r = a, the consistency requirement results in perturbations that are characterized by the condition u' = 0 at r = a [see Eqs. (25) and (34) of A66]. Imposing such a node on every axisymmetric mode of oscillation seems unphysical; it suggests that the boundary of the vortex core suppresses radial flow.

The present author sees no valid reason to reject a *formal* discontinuity of v' across r = a [cf. Arendt et al. 1997]. Physically, the discontinuity represents a rapid change across a thin transition layer. On the other hand, appendix A verifies that the radial eigenfunctions of u' and p' must be continuous at r = a for any normal mode of oscillation. Continuity of u' and p' at r = a is sufficient to derive an equation for the eigenfrequencies of all normal modes, after appropriate physical boundary conditions are applied at r = 0 and infinity [Kelvin 1880; Arendt et al. 1997]. It is here claimed that imposing continuity of v' at r = a is not merely superfluous, but incorrect.

The reader may question how the extra continuity constraint at r = a did not lead Ab-

dullah to an overdetermined problem. A solution was possible because the outer boundary condition was unspecified. The theory of A66 overlooks the requirement that acoustic waves created by the vortex must propagate outward as r tends toward infinity. The following subsection elaborates upon this oversight.

2.2 Radiation Condition

It is readily shown that the normal modes of A66 are inconsistent with the usual radiation condition for acoustic waves generated by the vortex. In A66, the (scaled) pressure perturbation associated with an axisymmetric normal mode is assumed to vary as

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$$p' = [A_1 J_0(\chi_o r) + A_2 Y_0(\chi_o r)] \cos(\omega t) Z(z), \quad r > a,$$
(5)

in which J_{ν} and Y_{ν} are Bessel functions of the first and second kinds, $\chi_o^2 \equiv \omega^2/c_o^2 - k_z^2$, $\omega \equiv 2\pi f$ is the angular frequency of the mode, k_z is (essentially) the vertical wavenumber of the mode, and Z is the vertical structure function of the mode. The coefficients A_1 and A_2 are constants, and ω is supposed to be sufficiently large to ensure that $\chi_o^2 > 0$. As $r \to \infty$, the Bessel functions in Eq. (5) become trigonometric functions [Abramowitz and Stegun 1972 (AS72)]; specifically, it can be shown that

$$p' \to \left\{ \frac{A_1}{2} \left[\cos(\omega t + \chi_o r - \pi/4) + \cos(\omega t - \chi_o r + \pi/4) \right] + \frac{A_2}{2} \left[\sin(\omega t + \chi_o r - \pi/4) - \sin(\omega t - \chi_o r + \pi/4) \right] \right\} \sqrt{\frac{2}{\pi \chi_o r}} Z(z).$$
(6)

The terms proportional to A_1 and A_2 in Eq. (6) contain inward propagating waves. This suggests that the A66 modes incorporate unphysical acoustic sources (or reflective walls) outside the vortex.

3. Can Axisymmetric Radiation Reveal Vortex Features?

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Having explained the apparent oversights of A66, what remains is to reevaluate the actual importance of axisymmetric vortex modes to the acoustic radiation field. To gain insight, let us consider a Rankine vortex in which \bar{v} satisfies Eq. (3) and $\bar{u} = \bar{w} = 0$. More than a century ago, Lord Kelvin published a classic paper on the vibrations of an incompressible Rankine vortex [Kelvin 1880].³ He showed that the vortex possesses a discrete set of axisymmetric eigenmodes of the form

$$p' = A J_0(\xi r_{<}) K_0(k_z r_{>}) e^{i(k_z z - \omega t)} + c.c.,$$
(7)

in which A is an arbitrary constant, k_z is positive by convention,

$$\xi \equiv \sqrt{\frac{4\Omega^2 - \omega^2}{\omega^2}} k_z,\tag{8}$$

and $r_{<}(r_{>})$ is the lesser (greater) of the radial coordinate r and the core radius a. As usual, K_{ν} is a modified Bessel function of the second kind, and c.c. denotes the complex conjugate of the term to its left. The eigenfrequencies are given by the following well-known dispersion relation:

$$\frac{1}{a\xi} \frac{J_1(a\xi)}{J_0(a\xi)} = -\frac{1}{ak_z} \frac{K_1(ak_z)}{K_0(ak_z)}.$$
(9)

Equation (9) has an infinite number of real solutions $\{\pm \omega_n\}$, in which $n \in \{1, 2, 3, \dots \infty\}$. By construction, the value of ξ increases with n, so that n may be viewed as a surrogate for the radial wavenumber. The solution curves for $\omega_n(k_z)$ fall on and below the n = 1 branch that is shown in Fig. 2.

Let us now suppose that the medium is compressible, but that the Mach number $M \equiv \Omega a/c_o$ is much less than unity. In this parameter regime, Eq. (9) should remain a valid

³More recent discussions of perturbation theory for incompressible columnar vortices can be found in Saffman [1992], Arendt et. al [1997], and Fabre et al. [2006].

approximation of the Kelvin mode dispersion relation. On physical grounds, a Kelvin mode can emit an acoustic wave into the environment only if

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$$\omega_n > \omega_a \equiv c_o k_z. \tag{10}$$

The right-hand side of Eq. (10) is the minimum frequency of acoustic radiation for a specific k_z .

Appendix B shows that for $ak_z \ll 1$, the dispersion relation for the fastest (n = 1) branch of the Kelvin modes reduces to

$$\omega_1 \approx 2\Omega a k_z / j_{0,1},\tag{11}$$

in which $j_{0,1} = 2.40483$ is the smallest zero of J_0 . It follows that $\omega_1/\omega_a \sim 2M/j_{0,1} \ll 1$ as $ak_z \to 0$. The dispersion diagram further illustrates that ω_1 asymptotes to the inertial frequency 2Ω as $ak_z \to \infty$. More to the point, $d\omega_1/dk_z$ decays toward zero with increasing k_z . Consequently, the axisymmetric Kelvin modes remain slower than acoustic waves for all k_z , and can not excite acoustic radiation.

It is reasonable to speculate that compressibility introduces faster oscillations of the vortex core that readily generate acoustic radiation [cf. A66]. On the other hand, the pertinent spectrum of fast, axisymmetric eigenmodes could merely represent free-space sound waves,⁴ modified very slightly by the vortex. In the latter scenario, the character of the radiation would depend much more on the initial perturbation, or forcing, than on the structure of the vortex. Indeed, a brief numerical survey will provide evidence of the following:

The axisymmetric component of acoustic radiation emitted by a columnar vortex does not carry a robust vortex signature.

The numerics will be based on a linearized model of the perturbation dynamics in a parameter regime relevant to the problem of tornado infrasound.

⁴In this paper, "free-space" refers to an unbounded atmosphere without a vortex.

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4. Numerical Investigation of Axisymmetric Radiation

4.1 Experimental Setup

The numerical experiments involve a columnar vortex in a uniform atmosphere. For simplicity, gravity and vertical stratification are neglected. The unperturbed vortex satisfies the following condition of cyclostrophic balance:

$$\frac{d\bar{\Pi}}{dr} = \frac{1}{\theta_o} \frac{\bar{v}^2}{r},\tag{12}$$

in which $\overline{\Pi}(r)$ is the equilibrium Exner function and θ_o is the constant equilibrium potential temperature. By definition,

$$\Pi \equiv c_p (p/p_o)^{R/c_p} \quad \text{and} \quad \theta \equiv \frac{c_p}{R} \frac{p}{\rho \Pi},$$
(13)

in which p_o is the ambient pressure, R is the gas constant, c_p is the isobaric specific heat, and ρ is the local mass density.⁵ The unperturbed radial velocity (\bar{u}) and vertical velocity (\bar{w}) are assumed to be zero.

In keeping with standard practice, let us consider perturbations of the form

$$[u', v', w', \theta', \Pi'] = \left[\hat{u}(r, t), \hat{v}(r, t), \hat{w}(r, t), \hat{\theta}(r, t), \hat{\Pi}(r, t)\right] e^{ik_z z} + c.c.$$
(14)

Such perturbations constitute a complete basis for a general axisymmetric disturbance. To

⁵In atmospheric modeling studies, Π and θ are traditional substitutes for pressure and entropy [e.g., Klemp and Wilhelmson 1978].

lowest order in amplitude, they are governed by the following linearized equations of motion:

$$\frac{\partial \hat{u}}{\partial t} = \frac{2\bar{v}}{r}\hat{v} - \theta_o \frac{\partial \hat{\Pi}}{\partial r} - \hat{\theta} \frac{d\bar{\Pi}}{dr}, \qquad \frac{\partial \hat{v}}{\partial t} = -\bar{\zeta}\hat{u} + F_v,$$

$$\frac{\partial \hat{w}}{\partial t} = -ik_z\theta_o\hat{\Pi}, \qquad \qquad \frac{\partial \hat{\theta}}{\partial t} = F_\theta,$$
(15)

$$\frac{\partial \hat{\Pi}}{\partial t} = -\hat{u}\frac{d\bar{\Pi}}{dr} - \frac{\bar{c}^2}{\theta_o}\frac{1}{r}\frac{\partial(r\hat{u})}{\partial r} - ik_z\frac{\bar{c}^2}{\theta_o}\hat{w} + \frac{\bar{c}^2}{\theta_o^2}F_{\theta}$$

in which $\bar{c}^2 \equiv (R/c_v)\theta_o \bar{\Pi}$ is the square of the sound speed, c_v is the isovolumic specific heat, and (as before) $\bar{\zeta} \equiv r^{-1}d(r\bar{v})/dr$ is the unperturbed axial vorticity. The symbol F_G denotes a forcing function of the variable \hat{G} .

The perturbation equations are solved numerically with low-order finite differencing in r, and fourth-order Runge-Kutta steps in t. The computational grid has 1 m increments and extends to r = 10 km. A linear sponge-ring absorbs outward propagating acoustic waves for $r \ge 8$ km. The values of θ_o and $c_o \equiv \bar{c}(\infty)$ are 300 K and 347.2 m s⁻¹, respectively.

For brevity, let us focus on cases where the basic state of the vortex has a regularized Rankine structure. Specifically, the vorticity profile is given by

$$\bar{\zeta}(r) = \Omega\left[1 - \tanh\left(\frac{r-a}{0.1a}\right)\right],\tag{16}$$

in which Ωa is approximately the maximum tangential windspeed, and a is approximately the radius of maximum wind. The forcing is either an extended thermal impulse, in which $F_v = 0$ and

$$F_{\theta} = \epsilon \eta \theta_o e^{-r^2/b^2} \times \begin{cases} 1 - \cos\left(2\pi t/\tau\right) & 0 < t < \tau, \\ 0 & t > \tau, \end{cases}$$
(17)

or an instantaneous angular impulse, in which $F_{\theta} = 0$ and

 $F_v = \epsilon \bar{v} \left[1 - \tanh\left(\frac{r-b}{0.1b}\right) \right] \delta(t).$ (18)

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The timescale τ (of F_{θ}) and radial lengthscale *b* are adjustable parameters. The dimensionless perturbation strength ϵ does not change from one experiment to another. The specific value of ϵ is arbitrary for the purpose of our discussion. The factor $\eta = 1 \text{ s}^{-1}$ is included in F_{θ} for dimensional consistency.

Table 1 lists the key parameters of each numerical experiment. A few of the experiments involve perturbations of free-space. These control experiments are labeled FS, followed by a lower-case letter. The remaining experiments involve perturbations of a modest vortex $(a = 100 \text{ m}, M = \Omega a/c_o = 0.14)$ or a formidable vortex (a = 300 m, M = 0.43). The modest and formidable vortices are named V1 and V2, and the experiments are labeled accordingly.

4.2 Disturbances with Moderate Vertical Wavelengths

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Figure 3 shows the power spectra of $\hat{\Pi}$ on the central axis (r = 0) and in the acoustic radiation zone (r = 2.3 km) for experiments FSa, V1a, V1d and V2a. In each case, the vertical wavelength of the perturbation $(\lambda_z \equiv 2\pi/k_z)$ is 100 m. The power spectra are calculated after applying Hann windows to 12 minute time series, starting from t = 0. Similar results have been verified for 3 minute time series.

The control experiment (FSa) involves a 1-s heat pulse applied to a resting atmosphere. Both power spectra in FSa have solitary peaks near $f = c_o/\lambda_z = \omega_a/2\pi$. This minimal acoustic frequency is characteristic of sound waves with $\lambda_z \ll \lambda_h$, in which λ_h is the horizontal wavelength. Such waves linger in the computational domain because of their infinite vertical extent and small horizontal group velocity, $c_{gh} = c_o/\sqrt{(\lambda_h/\lambda_z)^2 + 1}$.

²¹⁰ Unlike the FSa experiment, the central perturbation of V1 consistently has strong spectral ²¹¹ peaks for $f \ll c_o/\lambda_z$. The low-frequency peaks are prominent whether F_{θ} (in V1a) or F_v (in ²¹² V1d) generates the disturbance. Figure 4 verifies that the low-frequency peaks correspond to ²¹³ the Kelvin modes of an ideal Rankine vortex with the same parameters as V1 [cf. Eq. (9)]. ²¹⁴ As explained previously, axisymmetric Kelvin modes do not radiate; consequently, their spectral signatures are not seen in the radiation zone. In fact, the power spectrum of the outer acoustic perturbation in experiment V1a has the same form as its counterpart in experiment FSa.

The behavior of V2 is qualitatively similar to that of V1, as illustrated by the two experiments (V1a and V2a) in which a 1-s heat pulse generates the disturbance. In both cases, the heat pulse excites non-acoustic Kelvin modes and long-lived acoustic oscillations characterized by $\lambda_z \ll \lambda_h$. Although V2 has thrice the radius of V1, the spectral peak of the acoustic perturbation in experiment V2a does not budge from $f = c_o/\lambda_z$. In contrast to the prediction of A66, there is no clear evidence of a significant acoustic peak in which the central frequency varies as 1/a.

There is one notable difference between experiments V1a and V2a. Comparison of Figs. 3b and 3d suggests that a 1-s heat pulse excites the Kelvin modes of V1 more efficiently than those of V2. However, the excitation of Kelvin modes depends on the parameters ak_z and a/b. Both of these parameters differ by a factor of 3 between the two experiments under consideration.

4.3 Disturbances with Infinite Vertical Wavelengths

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Figure 5 displays time series of Π' in the acoustic radiation zone, for disturbances with infinite vertical wavelengths. Plots 5a and 5b show infrasonic emissions generated by distinct thermal forcing functions. Both plots suggest that V2 emits slightly weaker infrasound than V1 in response to F_{θ} , but emissions from neither vortex differ significantly from the infrasound generated by F_{θ} in free-space. Plot 5c shows infrasound generated by an angular impulse F_v . If the angular impulse scales in size and strength with the vortex, then the infrasonic emission reflects the size and strength of the vortex from which it comes. However, with the same angular impulse, V1 and V2 emit the same waves.⁶

⁶Note that the wave amplitudes are the same, because the cores of V1 and V2 have equivalent angular rotation frequencies. In linear theory [Eqs. (15)], the amplitude of the acoustic emission created by any

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4.4 Primary Inference

The author concludes that the form of axisymmetric radiation is primarily determined by the forcing function, not the characteristics of the vortex. Therefore, the axisymmetric component of vortex infrasound can not be used to estimate vortex size or strength with a high degree of confidence.

5. Closing Remarks

In contrast to the speculations of A66, it has been shown that axisymmetric radiation does not carry a robust signature of the vortex core. The axisymmetric Kelvin modes are undamped, but non-radiative. Moreover, the infrasound of an axisymmetric disturbance is shaped by the forcing that creates it. These conclusions were derived from an investigation of Rankine vortices, but seem more general. The author has found similar results for other centrifugally stable vortices with either monotonic or non-monotonic radial distributions of axial vorticity. One could hypothesize that growing axisymmetric perturbations of centrifugally unstable vortices are exceptional, and generate distinct infrasound. However, the considerations of appendix C suggest that if an axisymmetric vortex mode grows with time, it must be non-radiative.

There is no obvious reason (to the author) why the conclusions of this paper should not extend to helical vortices with nonzero \bar{w} . It is true that vertical flow introduces the possibility of new, axisymmetric Kelvin-Helmholtz (AKH) instabilities [Martin and Meiburg 1994; Rotunno 1978; cf. Batchelor and Gill 1962]. However, the characteristic frequency of an AKH mode does not exceed $c_o k_z$, if the axial Mach number is much less than unity. In other words, a typical AKH mode does not satisfy the radiation condition given by Eq. (10).

angular impulse F_v decays to zero with Ω . By contrast, several tests have shown negligible sensitivity of the emitted wave form (frequency) to variation of Ω in the range 0.2-2 s⁻¹, with *a* and F_v held unchanged.

On the other hand, it is well-known that *non-axisymmetric* Kelvin modes and shear-267 flow instabilities can readily generate infrasound [Powell 1964; Broadbent 1984; Sozou 1987; 268 Howe 2003; Roberts 2003]. The frequency of such infrasound is proportional to Ω , and 269 therefore carries some information about the vortex. For an ordinary tornado, $\Omega/2\pi$ is of 270 order 0.1 Hz. Theoretical studies suggest that diabatic processes in cloud turbulence can 271 be strong sources of infrasound in the very same 0.1 Hz frequency regime [Akhalkatsi and 272 Gogoberidze 2009,2011]. Therefore, the infrasound of non-axisymmetric Kelvin modes and 273 shear-flow instabilities may not be discernible under usual circumstances. This conclusion is 274 consistent with earlier field studies that found no clear relationship between severe weather 275 infrasound and tornadoes at frequencies below 0.2 Hz [Bowman and Bedard 1971; Georges 276 and Greene 1975]. Nevertheless, the provisional simulations of Schecter et al. [2008] suggest 277 that discernible signals generated by non-axisymmetric Kelvin modes might be possible under 278 exceptional circumstances, if the windspeed of the tornado greatly exceeds 50 m s⁻¹. 279

On empirical grounds, it seems reasonable to maintain that ordinary tornadoes cause abnormally strong levels of 0.5-10 Hz infrasound [Bedard 2005]. That being said, a convincing theoretical explanation remains absent. This paper exposed critical deficiencies of the A66 theory, which has received considerable attention in the literature. A more successful theory of tornado infrasound may require consideration of non-columnar structure, diabatic cloud processes [cf. Akhalkatsi and Gogoberidze 2009,2011; Schecter and Nicholls 2010; Schecter 2011], or even electrodynamics [cf. Schmitter 2010]. Suffice it to say, further investigation is necessary.

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A. Eigenfunction Matching Conditions at the Boundary of the Vortex Core

The axisymmetric eigenmodes of a columnar vortex are perturbations in which

$$\Pi' \equiv \frac{R\bar{\Pi}}{c_p\bar{p}}p' = \Phi(r)e^{i(k_z z - \omega t)} + c.c., \tag{A1}$$

and likewise for all other fluid variables. The eigenfunction Φ and eigenfrequency ω are generally complex, but the vertical wavenumber k_z is real.

Using Eqs. (15), with all forcing terms set to zero, it is readily shown that Φ must satisfy the following second-order ordinary differential equation:

$$\frac{d}{dr}\left(\frac{\bar{\mu}}{\sigma}\frac{d\Phi}{dr}\right) = -\frac{\bar{\mu}}{\omega^2}\chi^2\Phi,\tag{A2}$$

in which $\bar{\mu} \equiv r \bar{\Pi}^{c_v/R}$, $\sigma \equiv \omega^2 - 2\bar{v}\bar{\zeta}/r$, and $\chi^2 \equiv \omega^2/\bar{c}^2 - k_z^2$. Integrating Eq. (A2) from the origin to r yields

$$\frac{d\Phi}{dr} = -\frac{\sigma}{\bar{\mu}} \int_0^r dr \frac{\bar{\mu}}{\omega^2} \chi^2 \Phi.$$
 (A3)

Because the right-hand side of Eq. (A3) is finite, Φ must be continuous at any r. Therefore, the first matching condition at the radius a of the vortex core is

$$\Phi(a_{+}) = \Phi(a_{-}), \tag{A4}$$

in which

$$a_{\pm} \equiv \lim_{\varepsilon \to 0^+} a \pm \varepsilon. \tag{A5}$$

Equation (A3) also implies that

$$\frac{1}{\sigma} \frac{d\Phi}{dr}\Big|_{a_{+}} = \frac{1}{\sigma} \frac{d\Phi}{dr}\Big|_{a_{-}}.$$
(A6)

This result may be applied to vortices with continuous or discontinuous σ . For the latter

 $_{312}$ case, it is a jump formula for the radial derivative of Φ across a vanishingly small transition $_{313}$ layer.

The second matching condition [Eq. (A6)] has a simple physical interpretation. Letting

$$u' = U(r)e^{i(k_z z - \omega t)} + c.c., \tag{A7}$$

it can be shown [from Eqs. (15)] that

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$$U = -i\theta_o \frac{\omega}{\sigma} \frac{d\Phi}{dr}.$$
 (A8)

Therefore, the second matching condition amounts to continuity of the radial velocity perturbation: $U(a_+) = U(a_-)$.

B. Asymptotic Solutions for the Largest Eigenfrequency of Axisymmetric Kelvin Modes

Assume $ak_z \ll 1$ and $\omega \ll 2\Omega$. In this parameter regime, the dispersion relation for axisymmetric Kelvin modes [Eq. (9)] reduces to

$$y\frac{J_0(y)}{J_1(y)} = (ak_z)^2 \ln(ak_z),$$
(B1)

in which

$$y \equiv 2ak_z \Omega/\omega. \tag{B2}$$

Here, $K_{\nu}(ak_z)$ has been replaced with the leading-order term in its asymptotic expansion for small ak_z [AS72]. Given that $(ak_z)^2 \ln(ak_z) \ll 1$ and $y/J_1(y)$ is of order unity or greater, the smallest value of y (the largest value of ω) that can satisfy Eq. (B1) is near the first root of J_0 . In other words, min $[y] \approx j_{0,1}$, which is an alternative expression of Eq. (11).

Let us now consider the limit $ak_z \to \infty$. In approaching this limit, the Kelvin-mode

dispersion relation [Eq. (9)] reduces to

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 $y \equiv ak_z \sqrt{\frac{4\Omega^2 - \omega^2}{\omega^2}}.$ (B4)

(B3)

Here, $K_{\nu}(ak_z)$ has been replaced with the leading-order term in its asymptotic expansion for large ak_z [AS72]. Given that $(ak_z)^{-1} \ll 1$, the smallest value of y (the largest value of ω) that can satisfy Eq. (B3) is close to the first nonzero root of J_1 . In other words, $\min[y] \to j_{1,1} = 3.83171$, or equivalently,

 $\frac{1}{y}\frac{J_1(y)}{J_0(y)} = -\frac{1}{ak_z},$

$$\omega_1 \longrightarrow \frac{2\Omega}{\sqrt{1 + (j_{1,1}/ak_z)^2}}.$$
(B5)

Therefore, the fastest axisymmetric Kelvin mode acquires the inertial frequency 2Ω at large vertical wavenumbers. As one may readily deduce, the same is true for all axisymmetric Kelvin modes [e.g. Saffman 1992].

C. Non-Existence of Unstable, Axisymmetric, Radiative Vortex Modes

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Multiplying Eq. (A2) by Φ^* , and integrating from r = 0 to a distant radius r_b , where the vortex velocity field is negligible, yields the following:

$$\frac{\bar{\mu}}{\omega^2} \Phi^* \frac{d\Phi}{dr} \bigg|_{r=r_b} - \int_0^{r_b} dr \frac{\bar{\mu}}{\sigma} \left| \frac{d\Phi}{dr} \right|^2 = -\int_0^{r_b} dr \frac{\bar{\mu}\chi^2}{\omega^2} \left| \Phi \right|^2.$$
(C1)

Here it is tacitly assumed that the eigenmode is regular on the central axis of the vortex, which must be the case if the eigenmode can *independently* represent a physical perturbation.

The imaginary part of Eq. (C1) can be written

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$$2\omega_I \omega_R (\mathcal{B}_R - \mathcal{I}_1) = (\omega_R^2 - \omega_I^2) \mathcal{B}_I, \qquad (C2)$$

in which $\omega_R(\omega_I)$ is the real (imaginary) part of ω ,

$$\mathcal{I}_1 \equiv \int_0^{r_b} dr \bar{\mu} \left(\frac{1}{\left|\sigma\right|^2} \left| \frac{d\Phi}{dr} \right|^2 + \frac{k_z^2}{\left|\omega^2\right|^2} \left|\Phi\right|^2 \right) > 0, \tag{C3}$$

and

$$\mathcal{B} \equiv \mathcal{B}_R + i\mathcal{B}_I \equiv \frac{\bar{\mu}}{|\omega^2|^2} \Phi^* \frac{d\Phi}{dr} \bigg|_{r=r_b}.$$
(C4)

It will be said that the mode is "bounded" if $\mathcal{B} \to 0$ as $r_b \to \infty$.

Suppose that the eigenmode corresponds to an oscillation of the vortex core that emits outward propagating acoustic radiation. If the vortex oscillation (the acoustic source) grows exponentially with time, causality requires that the radiation field (the outer part of $r^{1/2}\Phi$) decays exponentially with increasing r. In other words, a radiative vortex mode must be bounded if $\omega_I > 0$.

Setting \mathcal{B}_R and \mathcal{B}_I to zero in Eq. (C2) yields $\omega_I \omega_R = 0$. It follows that bounded, axisymmetric eigenmodes must have purely real or purely imaginary eigenfrequencies. But a *radiative* vortex mode must have nonzero ω_R . One is led to conclude that unstable ($\omega_I > 0$), axisymmetric, radiative vortex modes can not exist.

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FIGURE CAPTIONS

FIG. 1. The cylindrical coordinate system, centered on a columnar vortex with an axisymmetric perturbation. The deformed cylinder represents a pressure isosurface of the vortex core.

FIG. 2. The ω - k_z dispersion curve of the fastest axisymmetric Kelvin mode (ω_1) in the limit $M \to 0$. The dotted lines show the minimum sound-wave frequency ($\omega_a \equiv c_o k_z = ak_z \Omega/M$) at M = 0.01 and M = 1. The dashed line (ω_1^0) is the leading-order asymptotic solution for ω_1 as $ak_z \to 0$ [see Eq. (11)].

FIG. 3. Power spectra of Π' (the pressure perturbation) in four numerical experiments. Plots (a), (b) and (d) correspond to perturbations generated by 1-s heating in experiments FSa, V1a and V2a, respectively. Plot (c) corresponds to the perturbation generated by an angular impulse in experiment V1d. The solid curves are obtained from time series at the center of the vortex (r = 0), whereas the empty circles are obtained from time series in the acoustic radiation zone (r = 2.3 km). The vertical wavelength of each disturbance is $\lambda_z = 100$ m. All power spectra are normalized to the peak value at r = 2.3 km in experiment FSa. [The solid curve in (d) was smoothed with a sliding average over an interval of .007 Hz, in order to remove small-scale features that were fragile to the details of its computation.]

FIG. 4. Low-frequency power spectrum of the perturbation Exner function (pressure field) on the central axis of the vortex in experiment V1d. The numerical label of each prominent peak is the frequency of the closest axisymmetric Kelvin mode of an ideal Rankine vortex, from ω_1 (far right) to ω_6 (far left). The relevant parameters are $\Omega = 0.5 \text{ s}^{-1}$ and $ak_z = 2\pi$. The power spectrum is normalized to its peak value in the plotted frequency range.

FIG. 5. Time series of the perturbation Exner function (pressure field) in the radiation zone (r = 2.3 km) in three sets of numerical experiments with infinite λ_z . (a) Emissions produced by a 1-s heat pulse in experiments Fsb, V1b and V2b. (b) Emissions produced by a 10-s heat pulse in experiments Fsc, V1c and V2c. (c) Emissions produced by an angular impulse in experiments V1e, V2d and V2e. In each plot, Π' is normalized to the peak value in experiment Fsb.

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TABLE 1. Key parameters of the numerical experiments.

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Figure 1: The cylindrical coordinate system, centered on a columnar vortex with an axisymmetric perturbation. The deformed cylinder represents a pressure isosurface of the vortex core.



Figure 2: The ωk_z dispersion curve of the fastest axisymmetric Kelvin mode (ω_1) in the limit $M \to 0$. The dotted lines show the minimum sound-wave frequency $(\omega_a \equiv c_o k_z = a k_z \Omega/M)$ at M = 0.01 and M = 1. The dashed line (ω_1^0) is the leading-order asymptotic solution for ω_1 as $a k_z \to 0$ [see Eq. (11)].



Figure 3: Power spectra of Π' (the pressure perturbation) in four numerical experiments. Plots (a), (b) and (d) correspond to perturbations generated by 1-s heating in experiments FSa, V1a and V2a, respectively. Plot (c) corresponds to the perturbation generated by an angular impulse in experiment V1d. The solid curves are obtained from time series at the center of the vortex (r = 0), whereas the empty circles are obtained from time series in the acoustic radiation zone (r = 2.3 km). The vertical wavelength of each disturbance is $\lambda_z = 100 \text{ m}$. All power spectra are normalized to the peak value at r = 2.3 km in experiment FSa. [The solid curve in (d) was smoothed with a sliding average over an interval of .007 Hz, in order to remove small-scale features that were fragile to the details of its computation.]



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simulation	forcing	<i>a</i> (m)	$\Omega a \ (m/s)$	λ_z (m)	τ (s)	<i>b</i> (m)
FSa	thermal		0	100	1	100
FSb	thermal		0	∞	1	100
FSc	thermal		0	∞	10	100
V1a	thermal	100	50	100	1	100
V1b	thermal	100	50	∞	1	100
V1c	thermal	100	50	∞	10	100
V1d	v-impulse	100	50	100		100
V1e	v-impulse	100	50	∞		100
V2a	thermal	300	150	100	1	100
V2b	thermal	300	150	∞	1	100
V2c	thermal	300	150	∞	10	100
V2d	<i>v</i> -impulse	300	150	∞		300
V2e	<i>v</i> -impulse	300	150	∞		100

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