Note on Analyzing Perturbation Growth in a Tropical Cyclone-Like Vortex Radiating Inertia-Gravity Waves

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Abstract

A method is outlined for quantitatively assessing the impact of inertia-gravity wave radiation on the multimechanistic instability modes of a columnar stratified vortex that resembles an intense tropical cyclone. The method begins by decomposing the velocity field into one part that is formally associated with sources inside the vortex and another part that is attributed to radiation. The relative importance of radiation is assessed by comparing the rates at which the two partial velocity fields act to amplify the perturbation of an arbitrary tracer field—such as potential vorticity—inside the vortex. Further insight is gained by decomposing the formal vortex contribution to the amplification rate into subparts that are primarily associated with distinct vortex Rossby waves and critical layer perturbations.
1. Introduction

Tropical cyclones may exhibit various asymmetric instabilities as their basic states freely evolve or adjust to changing environmental conditions. Such instabilities can give rise to commonly seen elliptical cores, polygonal eyewalls and mesovortices [Muramatsu 1986; Reasor et al. 2000; Kossin and Schubert 2001; Corbosiero et al. 2006; Montgomery et al. 2006; Hendricks et al. 2012]. They may also induce horizontal mixing processes that efficiently redistribute angular momentum and equivalent potential temperature [Schubert et al. 1999; Kossin and Eastin 2001; Hendricks and Schubert 2010]. The immediate consequence of asymmetric instability and mixing can be the slowdown of intensification or a reduction of maximum wind speed in the primary circulation of the vortex [Schubert et al. 1999; Naylor and Schecter 2014; cf. Rozoff et al. 2009]. The possible negative influence of asymmetric instabilities may factor into why three-dimensional (3D) cloud-resolving tropical cyclone models often yield moderately or slightly weaker storms than their axisymmetric counterparts [Yang et al. 2007; Bryan 2012; Persing et al. 2013; Naylor and Schecter 2014].

In short, there is reason to believe that the theory of tropical cyclone intensity cannot be fully detached from the theory of vortex instability.

There are several well-known mechanisms of asymmetric vortex instability that are potentially relevant to the behavior of intense tropical cyclones. Classical barotropic instability mechanisms include (1) the mutual amplification of phase-locked counter-propagating vortex Rossby waves in the vicinity of the eyewall [Levy 1965; Michalke and Timme 1967; Schubert et al. 1999], and (2) the mutual amplification of a vortex Rossby wave and the potential vorticity (PV) anomaly that it generates in a suitably conditioned critical layer [Briggs et al. 1970]. Another viable mechanism of asymmetric perturbation growth is the positive feedback of inertia gravity wave radiation on the vortex Rossby wave that is responsible for

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1It should be noted that regardless of vortical shear-flow instabilities, the enabling of asymmetric moist convection can alter the angular momentum fluxes and thermodynamics that regulate storm intensity, with various effects that may or may not be negative [e.g., Persing et al. 2013].
its excitation [e.g., Ford 1994; Plougonven and Zeitlin 2002; Schecter and Montgomery 2004 (SM04); Hodyss and Nolan 2008 (HN08); Park and Billant 2013]. Instabilities related to baroclinic vortex structure [Kwon and Frank 2005] and the transient growth of nonmodal perturbations [Nolan and Farrell 1999; Antkowiak and Brancher 2004] are also pertinent, but will not be considered explicitly in this note.

The dominant modes of instability can involve multiple mechanisms operating simultaneously [Menelaou et al. 2016 (M16)]. Under these circumstances, the role of each mechanism in destabilizing the vortex is difficult to assess without the right diagnostic. The main purpose of this note is to briefly present an alternative method to quantitatively compare the importance of inertia-gravity wave radiation to that of other processes in driving the growth of vortex perturbations. The method amounts to comparing the rates at which the perturbation to a tracer field inside the vortex—such as PV—is amplified by velocity fields attributed to radiation and to sources within the vortex itself. The seeds for such an analysis were planted in a qualitative discussion of vortex instability in section 1 of HN08. The following broadens the discussion and explicates our procedure for quantitatively assessing the nature of an instability.

2. A Simple Model Suitable for a Study of Complex Instabilities

Consider a barotropic vortex in gradient-wind and hydrostatic balance. Herein, we shall assume that 3D perturbations of the balanced state obey linearized hydrostatic primitive equations, simplified with a Boussinesq approximation. The Coriolis parameter $f$ and the static stability $N^2$ are given constant values, and viscosity is entirely neglected. The reader is referred to section 2 of M16 for a detailed description of the applicable perturbation equations and the method used for this study to computationally find the dominant modes of vortex instability. Additional discussion of the linear model can be found in SM04.

Analysis of the perturbation dynamics is facilitated by introducing a cylindrical coordi-
nate system that is coaligned with the central axis of the vortex. As usual, $r$ and $\varphi$ represent the radial and azimuthal coordinates. To simplify various equations, the vertical coordinate $z$ is chosen to be the pressure based pseudo-height of Hoskins and Bretherton [1972]. The variables $u$, $v$ and $w$ denote (in order) the radial, azimuthal and pseudo-vertical components of the vector velocity field $\mathbf{v}$. Field variables dressed with overbars and primes respectively represent equilibrium and time-dependent perturbation fields. Each fluid variable is the sum of its equilibrium and perturbation components, as exemplified by $v = \bar{v}(r) + v'(r, \varphi, z, t)$, in which $t$ is time.

Henceforth, we will assume that the unperturbed vortex features an off-center relative vorticity peak [Fig. 1a] similar to that found in the eyewall region of a strong tropical cyclone [Rogers et al. 2013]. We will further assume that the angular velocity of the vortex greatly exceeds $f$, and the azimuthal velocity is comparable to the characteristic speed of an internal gravity wave. Under the preceding conditions, the dominant modes of asymmetric instability may involve a pair of vortex Rossby waves on opposite sides of the relative vorticity peak, two critical layer perturbations with distinct signatures in the PV field, and an outward propagating spiral inertia-gravity wave. Figures 1b and 1c illustrate the horizontal structure of such a growing perturbation. Further discussion of this figure is deferred to section 4.

3. A Method for Analyzing Perturbation Growth

Consider a fluid tracer $q$ whose unperturbed distribution $\bar{q}$ depends only on radius $r$. In the absence of forcing and diffusion, the linearized tracer equation is

$$\frac{\partial q'}{\partial t} + \bar{\Omega} \frac{\partial q'}{\partial \varphi} = -u \frac{d\bar{q}}{dr},$$

(1)

in which $\bar{\Omega}(r) \equiv \bar{v}/r$ is the equilibrium angular rotation frequency. It is suitable for the
present study to let $q$ equal PV. In the hydrostatic Boussinesq approximation, the materially conserved PV is defined by $q \equiv (\zeta + f \hat{z}) \cdot \nabla (\partial \phi / \partial z)$, in which $\zeta \equiv \nabla \times u$, $u \equiv v - w \hat{z}$, $\phi$ is the geopotential, and $\nabla$ is the 3D gradient operator [SM04]. The linearized PV perturbation is $q' = N^2 \zeta' + \eta \partial^2 \tilde{\phi} / \partial z^2$, in which $\zeta \equiv \hat{z} \cdot \tilde{\zeta}$, $\eta \equiv \zeta + f$, and $N^2 \equiv \partial^2 \tilde{\phi} / \partial z^2$.

Multiplying both sides of Eq. (1) by $q'$ and averaging over the azimuth $\varphi$ yields

$$\frac{1}{2} \partial \langle (q')^2 \rangle \quad (2a)$$

in which $\langle \ldots \rangle$ is the averaging operator. In theory, the radial velocity perturbation $u'$ that advects $q$ inward and outward can be viewed as a sum of contributions from each identifiable component ($\alpha$) of the growing mode and fluid boundaries should they exist. In other words, the small-amplitude tendency equation for $\langle (q')^2 \rangle$ can be decomposed as follows:

$$\frac{1}{2} \partial \langle (q')^2 \rangle = - \sum_{\alpha} \langle u'_\alpha q' \rangle \frac{d\tilde{q}}{dr} \quad (2b)$$

Assuming that there exists a $u'_\alpha$ in the vortex core attributable to the outer radiation field, the relative magnitude of its anti-correlation with $q'd\tilde{q}/dr$ would quantify the importance of radiation in driving the local growth of $q'$.

The method for partitioning the velocity perturbation is neither straightforward nor unique. One conceivable approach for a Boussinesq fluid might start by expressing the 3D-nondivergent velocity perturbation $\mathbf{v}'$ as a Biot-Savart-like volume integral of the vector vorticity perturbation and its image located beyond physical boundaries [e.g., Saffman 1992]. One could then separate the volume integral into several parts associated with different regions of the fluid and a part associated with the boundary conditions. Another approach might begin by separating $\mathbf{v}'$ into balanced and unbalanced components. Further decomposition of the initial separation could involve partial velocities attributable to different flow structures, such as Rossby-like waves, critical layer perturbations and inertia-gravity waves. A preliminary concern would be the best choice for the mesoscale balance model.
The following explores the usefulness of a simpler partitioning scheme that is deemed reasonable for analyzing the 3D instability of a barotropic vortex. Both binary and multi-component decompositions of the velocity perturbation are considered. The former begins by separating the flow-domain into a cylindrical region containing the vortex core, and an exterior radiation zone [Fig. 2a]. The boundary radius \( R \) corresponds to the outermost turning point where the modal perturbation starts to locally exhibit the characteristics of an inertia-gravity wave [M16, appendix B]. The multicomponent decomposition begins similarly, but further divides the vortex region into annular subregions that contain distinct peaks of wave activity associated with either a vortex Rossby wave or a critical layer disturbance [Fig. 2b]. The horizontal velocity perturbation \( u' \) at arbitrary \( z \) is then partitioned into components that are formally generated by the perturbations of vertical vorticity \( (\zeta') \) and horizontal divergence \( (\sigma') \) in each section of the fluid [cf. Bishop 1996; Renfrew et al. 1997]. Importantly, each component of \( u' \) has a (2D) irrotational and nondivergent extension beyond the section containing its vortical and divergent sources [Fig. 2c], and therefore contributes to the stirring of external tracers such as PV. Further details are forthcoming.

Constructing the partial velocity field associated with each section of the fluid is a relatively simple matter. The procedure begins by expressing the horizontal velocity perturbation as the gradient of a scalar potential \( (\chi') \) added to the cross-gradient of a streamfunction \( (\psi') \). In other words, let

\[
\begin{bmatrix}
u' \\
\psi'
\end{bmatrix} = \begin{bmatrix}
-\frac{1}{r} \frac{\partial \psi'}{\partial \varphi} + \frac{\partial \chi'}{\partial r} \\
\frac{\partial \psi'}{\partial r} + \frac{1}{r} \frac{\partial \chi'}{\partial \varphi}
\end{bmatrix}.
\]

(3)

For compact flows in unbounded domains, \( \psi' \) and \( \chi' \) are unique up to arbitrary constants. The compactness condition essentially applies to the problem at hand, because the radiation field of a temporally growing normal mode decays exponentially with increasing radius.
Taking the 2D curl and divergence of Eq. (3) yields

\[
\left( \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right) \begin{bmatrix} \psi' \\ \chi' \end{bmatrix} = \begin{bmatrix} \zeta' \\ \sigma' \end{bmatrix},
\]

(4)
in which \( \zeta' \equiv \frac{\partial(rv')/\partial r - \partial u'/\partial \varphi}{r} \) and \( \sigma' \equiv \frac{\partial(ru')/\partial r + \partial v'/\partial \varphi}{r} \).

The Poisson equations for \( \psi' \) and \( \chi' \) are readily solved upon expanding each perturbation field \( (g') \) into an azimuthal Fourier series of the form

\[
g'(r,z,t) = \sum_{n=-\infty}^{\infty} g_n(r,z,t)e^{in\varphi}.
\]

Substituting the appropriate Fourier series into Eq. (4) leads to a set of independent second-order ordinary differential equations (ODEs) for \( \psi_n \) and \( \chi_n \) in the variable \( r \). The ODEs can be formally solved with a Green function technique. The result is

\[
\begin{bmatrix} \psi_n(r,z,t) \\ \chi_n(r,z,t) \end{bmatrix} = \int_0^\infty d\tilde{r} \tilde{r} G_n(r,\tilde{r}) \begin{bmatrix} \zeta_n(\tilde{r},z,t) \\ \sigma_n(\tilde{r},z,t) \end{bmatrix},
\]

(5)
in which

\[
G_n(r,\tilde{r}) \equiv \begin{cases} -\frac{1}{2|n|} \left( \frac{r_<}{r_>} \right)^{|n|} & n \neq 0, \\ \ln(r/\tilde{r}) \Theta(r - \tilde{r}) & n = 0. \end{cases}
\]

(6)
The notation \( r_< (r_> \) is used above to denote the lesser (greater) of \( r \) and \( \tilde{r} \). The Heaviside step function is defined such that \( \Theta(r - \tilde{r}) = 1 \) for \( \tilde{r} < r \) and 0 for \( \tilde{r} > r \). The Green function \( G_n \) defined by Eq. (6) enforces appropriate boundary conditions in which the velocities corresponding to \( \psi_n \) and \( \chi_n \) are non-infinite as \( r \) tends toward 0 or \( \infty \).
Taking the Fourier transform of Eq. (3) and using Eq. (5) yields

\[
\begin{bmatrix}
    u_n \\
    v_n
\end{bmatrix} = \int_0^\infty d\tilde{r}\tilde{r} \begin{bmatrix}
    \tilde{\gamma}_u^n \\
    \tilde{\gamma}_v^n
\end{bmatrix},
\]

(7)

in which

\[
\begin{bmatrix}
    \tilde{\gamma}_u^n \\
    \tilde{\gamma}_v^n
\end{bmatrix} \equiv \begin{bmatrix}
    -inG_n(r,\tilde{r})\zeta_n(\tilde{r},z,t) + \frac{\partial}{\partial r}G_n(r,\tilde{r})\sigma_n(\tilde{r},z,t) \\
    inrG_n(r,\tilde{r})\zeta_n(\tilde{r},z,t) + \frac{\partial}{\partial r}G_n(r,\tilde{r})\sigma_n(\tilde{r},z,t)
\end{bmatrix}.
\]

(8)

Decomposing the integral in Eq. (7) into segments associated with the various regions of the fluid depicted in Fig. 2 yields

\[
\begin{bmatrix}
    u_n \\
    v_n
\end{bmatrix} = \sum_\alpha \begin{bmatrix}
    u_{n\alpha} \\
    v_{n\alpha}
\end{bmatrix} \equiv \sum_\alpha \int_\alpha d\tilde{r}\tilde{r} \begin{bmatrix}
    \tilde{\gamma}_u^n \\
    \tilde{\gamma}_v^n
\end{bmatrix},
\]

(9)

in which \( \int_\alpha \) denotes integration over region \( \alpha \). For a generic disturbance, the partial velocity field ascribed to region \( \alpha \) amounts to the following sum over all azimuthal wavenumbers:

\[
(u'_\alpha, v'_\alpha) = \sum_n (u_{n\alpha}, v_{n\alpha}) e^{in\phi},
\]

in which \((u_{n\alpha}, v_{n\alpha})\) is given by the integral-summand on the far-right hand side of Eq. (9).

The normal modes of a barotropic vortex are single-wavenumber perturbations whose pertinent fields have the form

\[
\begin{bmatrix}
    u' \\
    v' \\
    \phi' \\
    \zeta' \\
    \sigma' \\
    q'
\end{bmatrix} = a \begin{bmatrix}
    U(r) \\
    V(r) \\
    \Phi(r) \\
    Z(r) \\
    D(r) \\
    Q(r)
\end{bmatrix} \Upsilon(z) e^{i(n\phi - \omega t)} + c.c.,
\]

(10)

in which \( \omega \equiv \omega_R + i\omega_I \) is a complex frequency, \( a \) is a complex amplitude, and \( c.c. \) denotes the
complex conjugate required under the working assumption that \( n \) or \( \omega_R \) is nonzero [SM04; M16]. Taking the vertical boundaries to be isothermal (\( \partial \phi' / \partial z = 0 \)) at \( z = 0 \) and \( h \), the vertical wavefunction is given by \( \Upsilon = \cos(kz) \), in which \( k \) is an integral multiple of \( \pi/h \) [ibid].

The subcomponents of \((u', v')\) are expressible as

\[
\begin{bmatrix}
  u'_\alpha \\
  v'_\alpha
\end{bmatrix}
= a
\begin{bmatrix}
  U_\alpha(r) \\
  V_\alpha(r)
\end{bmatrix}
\Upsilon(z) e^{i(n\varphi - \omega t)} + \text{c.c.},
\tag{11a}
\]

in which

\[
\begin{bmatrix}
  U_\alpha(r) \\
  V_\alpha(r)
\end{bmatrix}
= \int_\alpha d\tilde{r} \begin{bmatrix}
  -i \frac{\partial}{\partial \tilde{r}} G_n(r, \tilde{r}) Z(\tilde{r}) + \frac{\partial}{\partial r} G_n(r, \tilde{r}) D(\tilde{r}) \\
  \frac{\partial}{\partial \tilde{r}} G_n(r, \tilde{r}) Z(\tilde{r}) + \frac{i}{r} G_n(r, \tilde{r}) D(\tilde{r})
\end{bmatrix}
\tag{11b}
\]

by virtue of Eq. (9).

Substituting Eq. (10) for \( q' \) and Eq. (11a) for \( u'_\alpha \) into Eq. (2b) yields the following modal growth rate formula:

\[
\omega_I = \sum_\alpha \frac{-\Re[U_\alpha Q^*]}{|Q|^2} \frac{dq}{dr} \equiv \sum_\alpha \omega_{I\alpha}(r).
\tag{12}
\]

Each partial growth rate \( \omega_{I\alpha} \) corresponds to one-half the local rate of change of \( \langle (q')^2 \rangle \) resulting from the radial advection of the tracer \( q \) by the velocity field ascribed to \( \zeta' \) and \( \sigma' \) in region \( \alpha \) of the normal mode. The value of \( \omega_{I\alpha} \) varies with \( r \) but not with \( z \), owing to the barotropic structure of the unperturbed vortex. Note that the value of \( \omega_{I\alpha} \) is the same regardless of whether \( q \) is PV or an arbitrary passive tracer, since the relation \( Q = -iU(dq/dr)/(\omega - n\Omega) \) is general.

If the distributions of \( \zeta' \) and \( \sigma' \) in region \( \alpha \) fully and exclusively constituted those of a particular dynamical element of the modal perturbation— such as a vortex Rossby wave, critical layer disturbance or inertia-gravity wave— one might reasonably connect \( \omega_{I\alpha} \) to the destabilizing (or stabilizing) influence of that element. On the other hand, one should bear in mind that the foregoing condition can be satisfied under normal circumstances only in some approximate sense, regardless of how carefully the vortex is partitioned. For example,
the velocity fields of a vortex Rossby wave are traditionally obtained by inverting a localized
PV or pseudo-PV perturbation according to specific balance conditions. As such, the wave
distributions of $\zeta'$ and $\sigma'$ generally extend beyond the localized PV or pseudo-PV anomaly,
into regions that are formally ascribed to other perturbation elements. While the far-
reaching extensions of vorticity and divergence may be weak, they could be relevant to
slow instabilities.

One might also worry about the appropriateness of instantaneous attribution. Although
mathematically valid, the idea of attributing part of the velocity field within the vortex to
simultaneous sources in the outer radiation field may seem physically questionable, owing to
the finite propagation speed of inertia-gravity waves. That being said, the actual information
contained in this partial velocity field amounts to the normal component of $u' - u'_{vtx}$ at the
boundary between the vortex ($\alpha = vtx$) and the radiation zone ($\alpha = rad$). Such is evident
by noting that inside the vortex, $u'_{rad} = \nabla_h \theta$, in which $\nabla_h^2 \theta = 0$ subject to $\partial \theta / \partial r = u' - u'_{vtx}$ at $r = R$. Here we have let $\nabla_h$ denote the horizontal gradient operator. Based
on the preceding consideration, one might view $u'_{rad}$ within the vortex as a flow-adjustment
connected to inertia-gravity wave emission at the boundary, without envisioning external
sources and sinks. In practice, $u'_{rad}$ may be readily obtained from the difference $u' - u'_{vtx}$
without any additional computation. For this study, the preceding expression is cross-checked
by calculating $u'_{rad}$ with the appropriate Green function integral between $R$ and a sufficiently
large radius that ensures convergence within a very small fractional error.

As a final remark, over the bulk of the vortex region, the dominant modes of instability
considered herein are intrinsically slow relative to inertial oscillations [M16]; that is, the
magnitude of $\omega_R - n \bar{\Omega}$ is appreciably less than $[(2 \bar{\Omega} + f)(\bar{\zeta} + f)]^{1/2}$. The preceding condition
suggests that the perturbation dynamics within the vortex is quasi-balanced [Shapiro and
Montgomery 1993]. As such, attributing local partial velocity fields to instantaneous nonlo-
cal sources within the vortex or on its boundary seems consistent (in a general sense) with
normal practice and thinking.
4. Illustrative Implementation of the Method

For illustrative purposes, we consider the asymmetric normal modes of cyclonic vortices whose unperturbed relative vorticity distributions have the form

\[
\bar{\zeta} \equiv \zeta_0 \left\{ \frac{1}{1 + (r/r_v)^\Delta} - \frac{\beta}{1 + [r/(\mu r_v)]^\Delta} \right\}, \tag{13}
\]

in which \(0 < \mu < 1, 0 < \beta < 1, \Delta \gg 1, r_v\) approximates the radius of maximum wind speed, and \(\zeta_0\) is a positive scaling factor. The vorticity distribution defined by Eq. (13) possesses an off-center peak between \(\mu r_v\) and \(r_v\), whose edges become square as \(\Delta \to \infty\). Increasing the dimensionless parameter \(\beta\) enhances the central vorticity deficit. Figure 1a shows the particular distribution with \(\mu = 0.6, \beta = 0.8\) and \(\Delta = 25\), along with the corresponding angular velocity field \(\bar{\Omega}\). The modal instabilities are completely controlled by the variables shaping \(\bar{\zeta}\) (\(\mu, \beta, \Delta\)) and the following two dimensionless parameters:

\[
Ro \equiv \frac{2\bar{\Omega}_v}{f} \quad \text{and} \quad Fr \equiv \frac{\bar{v}_v}{N k^{-1}}, \tag{14}
\]

in which \(Ro\) is the Rossby number and \(Fr\) is a rotational Froude number based on the vertical wavenumber \(k\) of the disturbance. The \(v\)-subscripts on \(\bar{\Omega}\) and \(\bar{v}\) indicate that the variables are evaluated at \(r = r_v\). The perturbation depicted in Figs. 1b and 1c is the fastest growing \(n = 2\) normal mode of the vortex in Fig. 1a, with \(Ro = 100\) and \(Fr = 2.6\). It is equivalent to the normal mode appearing in Fig. 2 of M16.
The fluid partitioning sketched in Fig. 2b can be summarized as follows:

\[ \alpha \in \left\{ \begin{array}{ll}

\text{iw} : & 0 \leq r \leq r_c \text{ excluding icl}; \\
\text{icl} : & r_{*i}^{-} \leq r \leq r_{*i}^{+}; \\
o\text{w} : & r_c < r \leq R \text{ excluding ocl}; \\
o\text{cl} : & r_{*o}^{-} \leq r \leq r_{*o}^{+}; \\
\text{rad} : & r > R. \\
\end{array} \right. \] (15)

Here we have introduced notations for the inner critical radius \((r_{*i})\) and the outer critical radius \((r_{*o})\) of the instability mode, which represent the two solutions of

\[ n\Omega(r_{*}) = \omega_R. \] (16)

Moreover, we have let \(r_{*}^{\pm} = r_{*} \pm \delta r_{*}\), in which \(\delta r_{*} \equiv c \left| \omega_I/(nd\bar{\Omega}/dr) \right|_{r_{*}}\) is the nominal half-width of the linear critical layer [Schecter et al. 2000]. The constant \(c\) is taken to be 2 unless stated otherwise. The symbol \(r_c\) denotes the nonzero finite radius at which \(d\bar{\zeta}/dr = 0\), or \(r_{*i}^{+}\) if the latter is larger. The inner wave section (iw) is the central circle of radius \(r_c\), excluding the inner annular critical layer (icl). The outer wave section (ow) is the annulus between \(r_c\) and the inner boundary radius \(R\) of the radiation zone (rad), excluding the outer critical layer (ocl). Although the inner and outer wave sections (iw and ow) may each contain two disconnected regions separated by a critical layer, the former reduces to a single disc of radius \(r_{*i}^{-}\) when \(r_c = r_{*i}^{+}\). The vortex region (vtx) comprises all sections of the fluid but the radiation zone, and therefore covers the entire interval of \(r\) between 0 and \(R\).

Separating the vortex region from the radiation zone at the outermost turning point \(R\) of the instability mode seems relatively uncontroversial. The rationale for further decomposition of the vortex region requires additional discussion. To begin with, each subsection of the vortex region contains a distinct extremum of the angular pseudomomentum density of the instability mode. The angular pseudomomentum density is a standard measure of
local wave activity in systems with cylindrical geometry. Averaging over $\varphi$ and $z$, the
modal angular pseudomomentum density at any given time is proportional to the following
function [SM04, M16]:

$$\mathcal{L}(r) \equiv \mathcal{L}^{PV} + \mathcal{L}^{v\phi},$$

(17a)

in which

$$\mathcal{L}^{PV} \equiv -\frac{|a|^2 r^2 |Q|^2}{2d\bar{q}/dr} \quad \text{and} \quad \mathcal{L}^{v\phi} \equiv -|a|^2 k^2 r^2 \Re[V\Phi^*].$$

(17b)

Here, $Q$ and $\bar{q}$ are the perturbation wavefunction and basic state distribution of PV, as
opposed to a generic tracer. For all of the instability modes under consideration, $\mathcal{L}^{PV}$ tends
to dominate $\mathcal{L}^{v\phi}$ for $r < r^+_o$. Figure 1b is essentially a plot of $|\mathcal{L}^{PV}|^{1/2} \cos(n\varphi + \varphi_q + \varphi_a)$, in
which $\varphi_q$ ($\varphi_a$) is the phase of $Q$ ($a$). It is seen that each vortex section defined above [Eq. (15)]
contains a distinct peak of $|\mathcal{L}^{PV}|$. It has been verified with various diagnostics that the
peaks within the inner and outer wave sections of the vortex (near $\mu r_v$ and $r_v$) correspond
to counter-propagating vortex Rossby waves [M16, section 3b therein]. The peaks within
the inner and outer critical layers (near $r^+_i$ and $r^+_o$) are obviously generated by resonant
stirring of PV. Similar structure is found in all of the instability modes examined in this
note and in the more comprehensive study of M16. A caveat is that one or more of the
modal elements (such as the outer critical layer disturbance) may be negligible.

Figure 3a shows the radial variation of the two components of $\omega_I$ attributed to radiation
and internal vortex dynamics, for the instability mode appearing in Figs. 1b and 1c. The
top graph shows $\omega_{Irad}$ and $\omega_{Ivtx}$, while the bottom graph shows the aforementioned partial
growth rates multiplied by $|\mathcal{L}^{PV}|$. Also shown are $\omega_I$ and $\omega_I |\mathcal{L}^{PV}|$; whereas the former is a
constant, peaks in the latter correspond to regions of maximal vortex Rossby wave activity
and critical layer stirring. It is seen that $\omega_{Irad}$ considerably exceeds $\omega_{Ivtx}$ in regions of peak
vortex Rossby wave activity, whereas the opposite holds in the critical layers. In other words,
the horizontal velocity field attributed to radiation is primarily responsible for the growth of
$q'$ associated with the vortex Rossby waves, whereas the horizontal velocity field generated
by $\zeta'$ and $\sigma'$ within the vortex primarily controls the growth of $q'$ in the critical layers.

Figure 3b shows the radial variations of the four subcomponents of $\omega_{I_{vtx}}$. The subcomponents paint a more complex picture of the instability mode. Cancellations between the subcomponents account for the smallness of $\omega_{I_{vtx}}$ where the vortex Rossby wave activity is concentrated. The inner and outer wave regions are seen to generate velocity fields that act to amplify $q'$ in each other but mostly damp $q'$ locally. The velocity field produced by sources in the inner critical layer hinders the growth of $q'$ in both vortex Rossby waves. The velocity field produced by sources in the outer critical layer adds slightly to the damping effort in the outer wave. The positive vortex contribution to the growth of $q'$ in the inner critical layer is due to the positive influence of $u_i'$ exceeding the negative influence of $u_o'$. The growth of $q'$ in the outer critical layer is mostly due to the stirring induced by $u_o'$.

Additional information on the nature of the instability can be obtained by splitting each partial growth rate $\omega_{I\alpha}$ into subparts associated with $\zeta'$ and $\sigma'$ individually. That is, let

$$\omega_{I\alpha} = \omega_{I\alpha\zeta} + \omega_{I\alpha\sigma},$$

in which

$$\omega_{I\alpha s} \equiv \frac{-\Re[U_{\alpha s}Q^*]}{|Q|^2} \frac{d\bar{q}}{dr}$$

and

$$U_{\alpha s} = \begin{cases} \int_{\alpha} d\tilde{r} \frac{-in}{r} G_n(r, \tilde{r}) Z(\tilde{r}) & s = \zeta, \\ \int_{\alpha} d\tilde{r} \frac{\partial}{\partial r} G_n(r, \tilde{r}) D(\tilde{r}) & s = \sigma. \end{cases}$$

Figure 3c shows the radial variations of $\omega_{I\alpha\zeta}$ and $\omega_{I\alpha\sigma}$ for the instability mode at hand, with $\alpha \in \{vtx, rad\}$. Unsurprisingly, it is found that $\omega_{I\text{rad}\sigma} \gg \omega_{I\text{rad}\zeta}$. Less anticipated, one can see that $\omega_{I\text{vtx}\sigma}$ has values comparable and opposite to those of $\omega_{I\text{vtx}\zeta}$ in the regions of peak vortex Rossby wave activity.

The preceding growth rate decomposition [Fig. 3] is found to exhibit only moderate sensitivity to variations of $\delta r_*$, $r_c$ and $R$. Clearly, variations of $\delta r_*$ and $r_c$ that are constrained to prevent regional overlap have no bearing on the values of $\omega_{I_{vtx}}$ or $\omega_{I_{\text{rad}}}$. Variation of $\delta r_*$ from one-half to twice its standard value most notably coincides with a proportional increase in the growth rate of $q'$.
amplification of the ratio of $\omega_{I_{\text{icl}}}$ to $\omega_{I_{\text{iw}}}$ in the neighborhood of $\mu r_v$. Reduction of $r_c$ to $r_{s_1}^+$ decreases the positive magnitude of $\omega_{I_{\text{ow}}}$ by 37% (39%) at $\mu r_v (r_v)$. The corresponding local changes to $\omega_{I_{\text{iw}}}$ are equal in absolute value but opposite in sign. Reduction of $R$ to $r_{s_0}^+$ most notably increases the positive value of $\omega_{I_{\text{rad}}}$ by 26% at $r_v$, and commensurately intensifies the local negative value of $\omega_{I_{\text{vtx}}}$.

It is worth remarking that we have conducted a simple test to gain confidence that $\omega_{I_{\text{rad}}} \gg \omega_{I_{\text{vtx}}}$ implies the importance of radiation in driving the local growth of the PV perturbation in a tropical cyclone-like vortex. It is well known that a monotonic cyclone ($\beta = 0$) would be stable in the absence of inertia-gravity waves [Montgomery and Shapiro 1995]. Moreover, it is reasonably well established that the dominant mode of instability of a monotonic cyclone involves the positive feedback between a vortex Rossby wave at the edge of the potential vorticity core ($r = r_v$) and inertia-gravity wave radiation [Ford 1994; SM04]. We have verified that when $\beta = 0$, the condition $\omega_{I_{\text{rad}}} \gg \omega_{I_{\text{vtx}}}$ holds very well in the vicinity of $r_v$ for a number of vortices with $Ro \gg 1$ and $Fr < \sim 1$. In each case considered, $\Delta$ was made sufficiently large to prevent significant opposition to modal growth by PV stirring in the outer critical layer [SM04]. Note that the extremely opposite condition, $\omega_{I_{\text{rad}}} = 0$, agreeably holds for all nondivergent barotropic ($Fr, k = 0$) instabilities. In this limit, $\zeta'$ and $\sigma'$ vanish outside the vortex, and the integral expression for $U_{\text{rad}}$ [Eq. (11b)] in the definition of $\omega_{I_{\text{rad}}}$ [Eq. (12)] is clearly zero.

5. Comparison to Alternative Diagnostics

The tracer based instability analysis offers a perspective on the importance of inertia-gravity wave radiation that may not fully agree with tentative assessments gleaned from simpler diagnostics. Discrepancies primarily occur when more than one mechanism has substantial impact on the amplification of a perturbation.

Recently, Menelaou and coauthors [M16] provisionally assessed the importance of radia-
tion by examining its contribution to the wave activity budget of a growing mode. The wave activity of region $\alpha$ was defined by

$$W_\alpha \equiv \int_\alpha dr \mathcal{L} e^{2\omega t}.$$  \hspace{1cm} (19)

For modes in which $|W_{ow}| \gtrsim |W_{iw}|$, conservation of total wave activity was expressed in the form

$$\frac{dW_{ow}}{dt} = - \sum_{\alpha \neq ow} \frac{dW_\alpha}{dt}.$$  \hspace{1cm} (20a)

Substituting Eq. (19) into the left-hand side of Eq. (20a) and dividing through by $2W_{ow}$ yields

$$\omega_I = \sum_{\alpha \neq ow} \hat{\omega}_I \alpha,$$  \hspace{1cm} (20b)

in which $\hat{\omega}_I \alpha \equiv -(dW_\alpha/dt)/(2W_{ow})$. A large relative magnitude of $\hat{\omega}_{I\text{rad}}$ on the right-hand side of Eq. (20b) simply implies that amplification of the radiation field (possessing negative wave activity) has an important role in balancing the growth of positive outer vortex Rossby wave activity. One might tentatively infer from such a result that radiation has an important role in driving the instability, but rigorous justification of this conclusion generally requires supplemental analysis and reasoning. We note that in practice, the calculation of $\hat{\omega}_{I\text{rad}}$ is simplified by reducing the integral $d\mathcal{W}_{I\text{rad}}/dt$ to an equivalent algebraic expression proportional to the angular momentum flux at $R$ [M16].

Hodyss and Nolan [HN08] examined the radial distribution of the contribution

$$S_r \equiv -r \langle u'v' \rangle d\Omega/dr$$  \hspace{1cm} (21)

to the growth rate of kinetic energy in the asymmetric perturbation. They showed that $S_r$ is concentrated beyond the edge radius ($r_v$) of the vorticity distribution if the instability primarily involves the positive feedback of inertia-gravity wave radiation on an outer vortex Rossby wave that is responsible for its emission. By contrast, they found that $S_r$
is concentrated inward of \( r_v \), if the instability primarily involves the interaction of counter-propagating vortex Rossby waves. One might therefore speculate that the importance of radiation in driving the instability of a tropical cyclone-like vortex could be assessed simply by comparing the magnitudes of \( S_r \) inward and outward of \( r_v \).

Figure 4 presents the three diagnostics at issue for three selected instability modes of a cyclonic vortex with \( \mu = 0.8, \beta = 0.9, \Delta = 40 \) and \( \text{Ro} = 100 \). The top row corresponds to the dominant \( n = 2 \) instability when the Froude number \( \text{Fr} \) has a subcritical value of 0.8. As in M16, the term “subcritical” refers to the small-\( \text{Fr} \) parameter regime in which inertia-gravity wave radiation has minimal influence on the fastest growing wavenumber-\( n \) eigenmode of the linearized dynamical system. The middle row corresponds to the dominant \( n = 2 \) instability when \( \text{Fr} \) has a strongly supercritical value of 6. The bottom row corresponds to the dominant \( n = 2 \) instability when \( \text{Fr} \) has a transitional value of 3. Assuming constant \( N \), the Froude number may be viewed as a dimensionless vertical wavenumber or a dimensionless measure of vortex strength. Taking the former perspective with \( \bar{v}_v \) having a severe tropical cyclone value of 65 m s\(^{-1}\), the three depicted modes of instability would have vertical quarter-wavelengths \( (\pi/2k) \) of (top) 12.8 km, (middle) 1.7 km and (bottom) 3.4 km. Here it is assumed that \( N \) is adequately approximated by a dry tropospheric value of 0.01 s\(^{-1}\); a reduction of \( N \) due to moisture would increase the vertical lengthscale associated with each mode.

The diagnostics under consideration offer a consistent picture of the subcritical instability mode. The binary growth rate partitioning advocated herein [Fig. 4a] suggests that inertia-gravity wave radiation is much less relevant to the amplification of \( q' \) than sources of the velocity perturbation (\( \zeta' \) and \( \sigma' \)) inside the vortex. The wave activity based growth rate partitioning of M16 [Fig. 4b] consistently suggests that radiation has minimal impact.\(^2\) The \( S_r \) profile [Fig. 4c] indicates that kinetic energy is transferred from the mean shear flow to the asymmetric perturbation primarily in the “eyewall” (\( \mu < r/r_v < 1 \)). There is little

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\(^2\)The negative values of \( \hat{\omega}_{I\text{ocl}} \) and \( \hat{\omega}_{I\text{icl}} \) in Fig. 4b do not imply that both critical layer perturbations hinder the growth of \( q' \) in the vicinity of the outer vortex Rossby wave; the value of \( \omega_{I\text{ocl}} \) is found to be positive at \( r_v \) (not shown).
evidence of such transfer in the outer core \((r/r_v > 1)\), where under different circumstances enhancement of \(S_r\) might have reflected appreciable positive feedback from radiation.

The opposite picture is found for the strongly supercritical instability mode. The tracer based instability analysis [Fig. 4d] reveals a dominant partial growth rate attributable to inertia-gravity wave radiation in all pertinent regions of the vortex, except the outer critical layer. The wave activity based growth rate decomposition [Fig. 4e] yields \(\dot{\omega}_{Irad} \gg \dot{\omega}_{I\alpha}\) for all \(\alpha \neq \text{rad}\). The distribution of \(S_r\) [Fig. 4f] is concentrated in the outer core, as in the principal radiation-driven instabilities of monotonic vortices.

The transitional instability mode exemplifies how the three diagnostics under consideration can leave different impressions. The tracer based instability analysis [Fig. 4g] suggests that radiation is equally or more responsible for the amplification of \(q'\) than sources of the velocity perturbation inside the vortex. A notable exception is in the inner critical layer, where the velocity field generated by vortex sources prevails. The wave activity based growth rate decomposition [Fig. 4h] consistently suggests that radiation is relevant to the instability. However, the relation \(\dot{\omega}_{Irad} < \dot{\omega}_{I\text{lw}} + \dot{\omega}_{I\text{cl}} + \dot{\omega}_{I\text{ocl}}\) leaves the inconsistent overall impression that radiation is less important than internal vortex dynamics. The \(S_r\) profile [Fig. 4i] indicates that kinetic energy is transferred from the mean shear flow to the asymmetric perturbation in both the eyewall and the outer core of the vortex, with no obvious discrimination. It is unclear to the authors how one might confidently assess the relative importance of radiation to the instability from the information contained in \(S_r\). In contrast to the tracer based analysis, an assessment based solely on the location of where \(S_r\) is peaked \((r < r_v)\) might encourage one to believe that internal vortex dynamics has the leading role in driving the instability. Alternatively, comparing the inner \((r \leq r_v)\) and outer \((r > r_v)\) radial integrals of \(rS_r\) yields an ambiguous inner-to-outer ratio of 1.2.

Note that of the three diagnostics under consideration, only the tracer based analysis was expressly designed to isolate and quantify the relative importance of radiation in forcing the growth of a perturbation field within the vortex. There may be no rigorous justification
for having presumed that one could find distinct patterns in the wave activity budget or the $S_r$-distribution to reliably convey the same information. A limited search for such patterns was deemed worthwhile, because variants of the aforementioned diagnostics are commonly examined and simpler to calculate. However, the preceding analysis of the transitional instability mode suggests that relatively simple diagnostics may be inescapably ambiguous when more than one mechanism has appreciable influence on the growth of a perturbation.

6. Summary

This note has expounded a previously underdeveloped method for evaluating the relative importance of inertia-gravity wave radiation in driving the instability of a columnar vortex resembling a tropical cyclone. The procedure begins by dividing the fluid volume into vortex and radiation zones. The velocity perturbation is then decomposed into one part that is formally associated with sources ($\zeta'$ and $\sigma'$) inside the vortex and another part that is attributed to radiation. The importance of radiation is assessed by comparing the rates at which the two partial velocity fields act to amplify a tracer perturbation, denoted by the variable $q'$ and exemplified by the PV perturbation in the vortex core.

As illustrated in section 4, the foregoing instability analysis can be readily extended to see how different sources of the velocity perturbation residing within the vortex individually contribute to the amplification of $q'$. Sources deemed relevant include those found in distinct critical layers and regions of enhanced vortex Rossby wave activity. In principle, an extended analysis can be beneficial for elucidating the true intricacy of a multimechanistic instability.

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References


Figure 1: (a) Basic state relative vorticity ($\tilde{\zeta}$) and angular velocity ($\tilde{\Omega}$) distributions of a tropical cyclone-like vortex whose shape parameters [Eq. (13)] are printed on the upper-right corner of the graph. Both distributions are normalized to $\tilde{\Omega}_v \equiv \Omega(r_v)$. Note that the unshown PV of the basic state $[\tilde{q} = (\tilde{\zeta} + f)N^2]$ closely resembles $\tilde{\zeta}$. (b) The scaled PV perturbation $q' r / |d\tilde{q}/dr|^{1/2}$ at arbitrary $z$ of the fastest-growing $n = 2$ eigenmode, when the Rossby and Froude numbers [Eq. (14)] are respectively given by $Ro = 100$ and $Fr = 2.6$. The color scale is normalized to the peak value of the plotted field. In the same units, the contour values are $\pm [0.04, 0.4, 0.9]$. (c) The geopotential perturbation $\phi'$. Solid/dashed contours correspond to the following positive/negative values: $\pm [0.06, 0.13, 0.21, 0.5, 0.75, 0.95]$ times the peak magnitude of the plotted field. The boundary of the yellow circle of radius $R = 1.83$ separates the vortex region from the radiation zone. All lengths in all parts of this figure are in units of the core radius $r_v$. The eigenfrequency of the mode depicted in (b) and (c) is $\omega = (1.16 + 0.06i)\tilde{\Omega}_v$. 
Figure 2: (a) Binary decomposition of the fluid into regions associated with the vortex (vtx, black) and radiation (rad, white). (b) Further decomposition of the vortex region into parts associated with the inner vortex Rossby wave (iw, dark gray), the inner critical layer (icl, white), the outer vortex Rossby wave (ow, black) and the outer critical layer (ocl, light gray). (c) Sketch of the velocity fields associated with localized positive relative vorticity ($\zeta'_+\alpha'$) and negative divergence ($\sigma'_-\alpha'$) perturbations in an arbitrary region ($\alpha$) of the fluid.
Figure 3: Partial growth rates of the instability mode shown in Figs. 1b and 1c. (a) Top panel: binary decomposition of the total growth rate (dotted black) into one part associated with vorticity and divergence anomalies inside the vortex (blue) and another part attributed to radiation (red). Bottom panel: similar to top panel, but with the growth rates multiplied by $|\mathcal{L}_{PV}|$ and normalized to the peak value of $\omega_I |\mathcal{L}_{PV}|$. (b) Similar to the bottom panel of (a) but for partial growth rates attributed to velocity sources in the inner wave region (iw, solid orange), outer wave region (ow, solid black), inner critical layer (icl, dashed orange) and outer critical layer (ocl, dashed black). Their sum (vtx, light blue) is shown for reference. (c) Similar to the bottom panel of (a), but with the partial growth rates attained from velocity sources in the vortex (blue) and in the radiation zone (red) each split into those attained from vorticity ($\zeta'$, solid) and divergence ($\sigma'$, dashed). Their sum (dotted black) is shown for reference.
Figure 4: (a-c) Instability diagnostics for the fastest-growing \( n = 2 \) eigenmode when \( \mu = 0.8, \beta = 0.9, \Delta = 40, \text{Ro} = 100 \) and \( \text{Fr} = 0.8 \). (a) Partial growth rates \( \{ \omega_{I\alpha} \} \) attributed to vorticity and divergence anomalies inside the vortex (vtx, blue) and to radiation (rad, red). Each is multiplied by \( |L^{PV}| \) and then normalized to the maximum of \( \omega_I |L^{PV}| \). Their scaled sum (dotted-black) is shown for reference. (b) The alternative partial growth rates \( \{ \hat{\omega}_{I\alpha} \} \) of M16. Those associated with the inner vortex Rossby wave (iw), inner critical layer (icl) and outer critical layer (ocl) are stacked in the blue column. The red column shows the formal contribution to \( \omega_I \) from radiation. (c) The production rate of kinetic energy in the \( n = 2 \) perturbation associated with the radial shear of \( \bar{\Omega} \), normalized to its maximum value. (d-f) As in (a-c) but when \( \text{Fr} = 6.0 \). (g-i) As in (a-c) but when \( \text{Fr} = 3.0 \). The turning points not shown alongside other important radii in (a), (d) and (g) respectively occur at \( R = 9.17, 15.54 \) and 18.80 in units of \( \bar{r}_v \). The eigenfrequencies of the top, middle and bottom modes are respectively \( \omega = 1.59 + 0.06i, 1.35 + 0.07i \) and \( 1.15 + 0.10i \) in units of \( \bar{\Omega}_v \).