Compressible f-Plane Solutions to Local Body Forces and Heatings; Part I: Initial Value and Forced/Heated Solutions

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Abstract

We determine the analytic, linear, f-plane, compressible solutions that arise in response to local, 3D, horizontal and vertical body forces and heatings in an isothermal, non-dissipative atmosphere. These body forces and heatings turn on and off smoothly over a finite interval in time, χ, with n oscillations occurring during this time. The solutions include mean flow, gravity waves (GWs), and acoustic waves (AWs). We find that the compressible solutions are different from the Boussinesq solutions for GWs with $\lambda_z \gtrsim 2\pi\mathcal{H}$. In the lower atmosphere, this is $\lambda_z \gtrsim 50$ km. As with the Boussinesq solutions, the compressible solutions also contain a cut-off factor which significantly reduces the amplitudes of GWs and AWs with frequencies ω less than $\hat{a}$ by $(2\pi)^{-1}\hat{a}^3/\omega^3$, where $\hat{a} = 2\pi n/\chi$. We also derive the compressible f-plane polarization relations for GWs and AWs. We then show that the phase relationships between the horizontal and vertical velocity, density, and temperature perturbations changes significantly when $|\lambda_z \gtrsim 2\pi\mathcal{H}|$.

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1. Introduction

GWs excited by wind flow over mountains and deep convection can excite GWs. Some of these "primary" GWs can propagate to the mesosphere, where they break. This saturation process creates horizontal body forces, which close the jets near the mesopause that exist if the atmosphere were in thermal equilibrium (see Fritts and Alexander, 2003 (hereafter FA03), and references therein). They also create a mean wind in the direction of GW propagation. This cools the summer mesopause because of an induced residual circulation (Nastrom et al., 1982; Holton, 1982, 1983; Dunkerton, 1982; Garcia, 1987). These body forces also radiate "secondary" GWs (Dickinson, 1969; Blumen, 1972; Walterscheid and Boucher, 1984; Zhu and Holton, 1987; Fritts and Luo, 1992; Luo and Fritts, 1993; Medvedev and Gavrilov, 1995). They are dubbed "secondary" because they arise in response to the breaking of "primary" waves.

Local, interval body forces and heatings describe the effects of outside processes on a local fluid. Here "local" describes a localized region in space, and "interval" describes a finite duration in time. These forcings/heatings can describe many of the effects of wave breaking, wave dissipation, convective overshoot, auroral heating, and geostrophic adjustment, to name a few. These forcings/heatings excite GWs and AWs, and may create mean flows. Both the temporal and spatial characteristics of the forcings/heatings determine the wave responses (Vadas and Fritts, 2001, hereafter VF01; Vadas *et al.*, 2003).

Although solutions have been derived for localized, interval heatings and body forcings (VF01), these are only applicable in the Boussinesq approximation (i.e., $\lambda_z \ll 4\pi\mathcal{H}$). However, compressible effects are expected to be important 1) for forcings/heatings deeper than the density scale height $\mathcal{H}$, and 2) for GWs with $\lambda_z \gtrsim \pi\mathcal{H}$. These latter GWs penetrate deeply into the thermosphere (Vadas, 2007; Fritts and Vadas, 2008), create large mean winds upon dissipating there (Vadas and Liu, 2009, hereafter VL09), and may seed plasma instabilities (Kelley, 1989; Takahashi *et al.*, 2009; Makela,*et al.*, 2010). AWs also play a role in the variability of the thermosphere (Lastovicka, 2006). Therefore, it is important to derive the analogous compressible solutions.

This paper is the first paper in a 2-paper series. The purpose of this paper is to derive the f-plane, compressible solutions to local interval body forcings/heatings. In doing so, we will generalize the VF01 solutions to include compressibility in full. We organize this paper as follows. Sec. 2 describes the f-plane, compressible fluid equations and solution methods. Sec. 3 derives the initial value solutions. The body forcing and heating solutions are derived in Sec. 4. Sec. 5 compares the compressible and Boussinesq solutions when $f = 0$. In Sec. 6, we calculate the exact, compressible, f-plane polarization relations for GWs and AWs. We then examine the phase relationships between the GW components when $\lambda_z \gtrsim 2\pi\mathcal{H}$. Sec. 7 contains our conclusions. Appendices A and B contain special case solutions to the forced/heated equations.

2. Compressible Solutions to Body Forces and Heatings

1. Linear, Compressible, f-plane Fluid Equations

We consider a 3D, local, interval body forcing/heating in our atmosphere which turns on and off smoothly over a finite interval in time. (For example, the interval could be 10 min or 6 hours, and the region could be $20^2 \times 10$ km^3 or $500^2 \times 5$ km^3 for convective overshoot or geostrophic adjustment, respectively.) This is a generalization of impulsive forcings/heatings, which turn on and off instantaneously in time. We assume that the fluid is inviscid. The compressible momentum, mass and energy conservation equations which describe this fluid are

$$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} + \frac{1}{\rho}\nabla p - \mathbf{g} + 2\mathbf{\Omega} \times \mathbf{v} = \mathbf{F}(\mathbf{x})\mathcal{F}(t) \tag{1}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla.\mathbf{v} = 0 \tag{2}$$

$$\frac{\mathrm{D}T}{\mathrm{D}t} + (\gamma - 1)T\nabla.\mathbf{v} = J(\mathbf{x})\mathcal{F}(t), \tag{3}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + (\mathbf{v}.\nabla), \tag{4}$$

$\mathbf{v} = (u, v, w)$ is the velocity vector in geographic coordinates, u, v, and w are the zonal, meridional, and vertical velocities, respectively, T is the temperature, ρ is the density, p is the pressure, Ω is the Earth's rotation vector, $\mathbf{g}$ is the Earth's gravitational force, $p = \mathrm{r}\rho T$ is the ideal gas law, $\mathrm{r} = 8308/\mathrm{X_{MW}}\ \mathrm{m^2\ s^{-2}\ K^{-1}}$, $\mathrm{X_{MW}}$ is the mean molecular weight of the particle in the gas, $\gamma - 1 = \mathrm{r}/C_v$, and C_v is the mean specific heat at constant volume. The body forcings/heatings have time dependence $\mathcal{F}(t)$, and occur only during the finite interval in time, χ. $\mathcal{F}$ and $d\mathcal{F}/dt$ must be continuous, analytic functions. The spatial portion of the 3D body force is $\mathbf{F}(\mathbf{x}) = F_x(\mathbf{x})\hat{\imath} + F_y(\mathbf{x})\hat{\jmath} + F_z(\mathbf{x})\hat{k}$, where $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are the zonal, meridional, and vertical unit vectors, respectively. The spatial portion of the heating is $J(\mathbf{x})$. These spatial distributions can be any arbitrary functions of (x, y, z) (although they and their spatial derivatives must be continuous functions). The body forcings and heatings, $\mathbf{F}(\mathbf{x})\mathcal{F}(t)$ and $J(\mathbf{x})\mathcal{F}(t)$, respectively, are hereafter called "sources", since they excite GWs and AWs, and create mean responses.

Using the ideal gas law, we express the energy equation, Eq. (3), in terms of pressure:

$$\frac{\mathrm{D}p}{\mathrm{D}t} + \gamma p\nabla.\mathbf{v} = \frac{p}{T}J(\mathbf{x})\mathcal{F}(t). \tag{5}$$

We can also express this in terms of the potential temperature, which is defined as

$$\theta = T(p_s/p)^{(\gamma-1)/\gamma}, \tag{6}$$

where p_s is standard pressure. Therefore,

$$\frac{1}{\theta}\frac{\mathrm{D}\theta}{\mathrm{D}t} = \frac{1}{T}\frac{\mathrm{D}T}{\mathrm{D}t} - \left(\frac{\gamma - 1}{\gamma}\right)\frac{1}{p}\frac{\mathrm{D}p}{\mathrm{D}t}. \tag{7}$$

Plugging in Eqs. (3) and (5), the $\nabla.\mathbf{v}$ terms cancel, yielding

$$\frac{1}{\theta}\frac{\mathrm{D}\theta}{\mathrm{D}t} = \frac{1}{\gamma T} J(\mathbf{x})\mathcal{F}(t). \tag{8}$$

Therefore, our conservation of energy equation (Eq. (3), (5), or (8)), expresses the well-known result that the potential temperature is conserved along the Lagrangian path of a fluid particle in a non-dissipative fluid if the external heat added is zero (e.g., Holton, 1992). We note that our compressible equation set, Eqs. (1)-(3), agrees with the compressible equations used in FA03 (Eqs.(1)-(5) in that work).

We now expand the fluid variables as background means plus perturbations:

$$\begin{aligned} u &= \overline{U} + u', \qquad v = \overline{V} + v', \qquad w = w', \\ \rho &= \overline{\rho} + \rho', \qquad T = \overline{T} + T', \qquad p = \overline{p} + p', \end{aligned} \tag{9}$$

where the overlines denote background mean values (which are constant in x, y, and t), and the primes denote perturbations. Thus the mean background wind is $\overline{U}\hat{\imath} + \overline{V}\hat{\jmath}$. We neglect the Earth's curvature; this limits horizontal wave scales to $\lambda_H \lesssim 20000 - 30000$ km. We also assume that the background fluid is isothermal ($\overline{T} = \overline{T}_0 = constant$ in z) and unsheared ($\overline{U}$ and $\overline{V}$ constant in z) in order to solve this problem analytically. Although beyond the scope of this paper, the excited waves can then be ray traced through wind shears, non-isothermal, and dissipative atmospheres (e.g., Vadas *et al.*, 2009a,b; VL09). Our goal here is to provide simple and insightful solutions of the mean, GW, and AW responses in an idealized background, but not the propagation and/or dissipation of these excited waves.

We assume that the fluid obeys the f-plane approximation: $2\mathbf{\Omega} \times \mathbf{v}' \simeq f(-v'\hat{\imath} + u'\hat{\jmath})$, where $f = 2\Omega \sin\theta$ and θ is latitude. This is a good assumption at mid and high latitudes. At $\theta = 45^{\circ}$ (mid) and $\theta = 75^{\circ}$ (high) latitudes, $f = 1.0 \times 10^{-4}$ s^{-1} and 1.4×10^{-4} s^{-1}, respectively. From Eq. (1), the hydrostatic equation (obtained when the perturbation quantities are set to zero) is

$$d\overline{p}/dz = -g\overline{\rho}, \tag{10}$$

where $\mathbf{g} = -g\hat{k}$. Solving Eq. (10), $\overline{\rho}$ and $\overline{p}$ decrease exponentially in z (Hines, 1960):

$$\overline{\rho} = \overline{\rho}_0 \mathrm{e}^{-z/\mathcal{H}}, \qquad \overline{p} = \overline{p}_0 \mathrm{e}^{-z/\mathcal{H}}, \tag{11}$$

where $\mathcal{H} = -\overline{\rho}(d\overline{\rho}/dz)^{-1} = \mathrm{r}\overline{T}_0/g$ is the density scale height, and $\overline{\rho}_0$ and $\overline{p}_0$ are the mean density and pressure at $z = 0$, respectively. We assume that the forcing/heating amplitudes are small enough that wave-mean flow and wave-wave interactions can be neglected. Linearizing Eqs. (1)-(2) and (5), we obtain:

$$\frac{\partial \mathbf{v}'}{\partial t'} + \frac{1}{\overline{\rho}}\nabla p' - \frac{\rho'}{\overline{\rho}^2}\nabla\overline{p} + f(-v'\hat{i} + u'\hat{j}) = \mathbf{F}(\mathbf{x})\mathcal{F}(t) \tag{12}$$

$$\frac{\partial \rho'}{\partial t'} + (\mathbf{v}'.\nabla)\overline{\rho} + \overline{\rho}\nabla.\mathbf{v}' = 0 \tag{13}$$

$$\frac{\partial p'}{\partial t'} + \mathbf{v}'.\nabla\overline{p} + \gamma\overline{p}\nabla.\mathbf{v}' = \frac{\overline{p}}{\overline{T}}J(\mathbf{x})\mathcal{F}(t), \tag{14}$$

where

$$\frac{\partial}{\partial t'} = \left(\frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial x} + \overline{V}\frac{\partial}{\partial y}\right). \tag{15}$$

In order for the coefficients on the left hand sides of Eqs. (12)-(14) to be constant with altitude, we define the following variables:

$$\begin{aligned}
\xi &= \mathrm{e}^{-z/2\mathcal{H}}u', \qquad \sigma = \mathrm{e}^{-z/2\mathcal{H}}v', \qquad \eta = \mathrm{e}^{-z/2\mathcal{H}}w', \\
\phi &= \mathrm{e}^{z/2\mathcal{H}}\rho'/\overline{\rho}_0 = \mathrm{e}^{-z/2\mathcal{H}}\rho'/\overline{\rho}, \qquad \psi = \mathrm{e}^{z/2\mathcal{H}}p'/\overline{p}_0 = \mathrm{e}^{-z/2\mathcal{H}}p'/\overline{p}, \\
\zeta &= \mathrm{e}^{-z/2\mathcal{H}}T'/\overline{T}_0.
\end{aligned} \tag{16}$$

Because u', v', w', $\rho'/\overline{\rho}$, $p'/\overline{p}$, and $T'/\overline{T}$ increase exponentially with altitude, the GW and AW amplitudes increase exponentially with altitude, as is well-known (Hines, 1960). We also scale the forcings and heatings as

$$
\begin{aligned}
F_{xs} &= \mathrm{e}^{-z/2\mathcal{H}} F_x, \qquad F_{ys} = \mathrm{e}^{-z/2\mathcal{H}} F_y, \\
F_{zs} &= \mathrm{e}^{-z/2\mathcal{H}} F_z, \qquad J_s = \mathrm{e}^{-z/2\mathcal{H}}\ \mathrm{r} J,
\end{aligned} \tag{17}
$$

where the subscript "s" denotes that the forcing/heating is scaled by $\mathrm{e}^{-z/2\mathcal{H}}$ in altitude.

We expand ξ, σ, η, ϕ, ψ, ζ, F_{xs}, F_{ys}, F_{zs}, and J_s in a Fourier series, e.g.,

$$
\xi(x,y,z,t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-i(kx+ly+mz)}\ \widetilde{\xi}(k,l,m,t) dk\ dl\ dm, \tag{18}
$$

where "$\sim$" denotes the Fourier transform of the variable, $\mathbf{k} = (k,l,m)$ is the wavenumber vector, and k, l, and m are the zonal, meridional, and vertical wavenumber components, respectively. Eqs. (12)-(14) become

$$
\frac{\partial \widetilde{\xi}}{\partial t''} - ik\widetilde{\psi} - f\widetilde{\sigma} = \widetilde{F_{xs}} \mathcal{F}(t) \tag{19}
$$

$$
\frac{\partial \widetilde{\sigma}}{\partial t''} - il\widetilde{\psi} + f\widetilde{\xi} = \widetilde{F_{ys}} \mathcal{F}(t) \tag{20}
$$

$$
\frac{\partial \widetilde{\eta}}{\partial t''} - im_s\widetilde{\psi} + g\widetilde{\phi} = \widetilde{F_{zs}} \mathcal{F}(t) \tag{21}
$$

$$
\frac{\partial \widetilde{\phi}}{\partial t''} - i\left(k\widetilde{\xi} + l\widetilde{\sigma} + m_s\widetilde{\eta}\right) = 0 \tag{22}
$$

$$
\frac{\partial \widetilde{\psi}}{\partial t''} + \delta\widetilde{\eta} - ic_s^2(k\widetilde{\xi} + l\widetilde{\sigma}) = \widetilde{J_s} \mathcal{F}(t), \tag{23}
$$

where the sound speed is c_s,

$$
c_s^2 = \gamma g \mathcal{H} = \gamma \mathrm{r} \overline{T}_0, \tag{24}
$$

$$
\delta = g(\gamma - 1) - ic_s^2 m_s, \tag{25}
$$

$$
m_s = m - i/2\mathcal{H}, \tag{26}
$$

$$
\frac{\partial}{\partial t''} = \frac{\partial}{\partial t} - i(k\overline{U} + l\overline{V}). \tag{27}
$$

The linearized ideal gas law is $p'/\overline{p} = \rho'/\overline{\rho} + T'/\overline{T}$. The scaled temperature perturbation is then

$$
\widetilde{\zeta} = \frac{\gamma}{c_s^2}\widetilde{\psi} - \widetilde{\phi}. \tag{28}
$$

Table 1 shows some important symbols used in this paper.

2. Solution methods

We take the Laplace transform of Eqs. (19)-(23). For example, the Laplace transform of $\widetilde{\psi}$, denoted $\mathcal{L}_{\widetilde{\psi}}$, is the integral (Abramowitz and Stegun, 1972)

$$\mathcal{L}_{\widetilde{\psi}} = \mathcal{L}(\widetilde{\psi}) = \int_0^\infty \mathrm{e}^{-s_r t}\widetilde{\psi} dt. \tag{29}$$

Therefore,

$$\mathcal{L}(\widetilde{\psi}_t) = s_r \mathcal{L}(\widetilde{\psi}) - \widetilde{\psi}(0), \tag{30}$$

where $\widetilde{\psi}(0)$ is the value of $\widetilde{\psi}$ at $t = 0$. We define

$$s = s_r - i(k\overline{U} + l\overline{V}). \tag{31}$$

The Laplace transform of the scaled pressure perturbation is then found to be

$$\begin{aligned} \mathcal{L}_{\widetilde{\psi}} \;=\; & \frac{1}{s(s^2 - s_1^2)(s^2 - s_2^2)} \Big\{ \quad \Big[(s^2 + i m_s g)(s^2 + f^2)\widetilde{\psi}(0) \\ & -\delta(s^2 + f^2)(s\widetilde{\eta}(0) - g\widetilde{\phi}(0)) + i c_s^2(\alpha s + \beta f)(s^2 + N_B^2)\Big] \\ & + \Big[i c_s^2(s A_\mathrm{F} + f B_\mathrm{F})(s^2 + N_B^2) - \delta s(s^2 + f^2)\widetilde{F_{zs}} \\ & +(s^2 + i m_s g)(s^2 + f^2)\widetilde{J_s}\Big] \mathcal{L}_\mathcal{F} \quad \Big\}. \end{aligned} \tag{32}$$

Here, the Laplace transform of $\mathcal{F}(t)$ is $\mathcal{L}_\mathcal{F}$, the buoyancy frequency is N_B, and

$$N_B^2 = (\gamma - 1)g^2/c_s^2 \tag{33}$$

$$\alpha = k\widetilde{\xi}(0) + l\widetilde{\sigma}(0) \tag{34}$$

$$\beta = k\widetilde{\sigma}(0) - l\widetilde{\xi}(0) \tag{35}$$

$$A_\mathrm{F} = k\widetilde{F_{xs}} + l\widetilde{F_{ys}} \tag{36}$$

$$B_\mathrm{F} = k\widetilde{F_{ys}} - l\widetilde{F_{xs}}. \tag{37}$$

Additionally, Eq. (32) contains s_1 and s_2, which are the 2 roots of the expression that result when deriving Eq. (32):

$$s^4 + [f^2 + c_s^2(\mathbf{k}^2 + 1/4\mathcal{H}^2)]s^2 + c_s^2[k_H^2 N_B^2 + f^2(m^2 + 1/4\mathcal{H}^2)] = 0. \quad (38)$$

Here, $\mathbf{k}^2 = k^2 + l^2 + m^2$ and $k_H^2 = k^2 + l^2$. Defining

$$a = -(s_1^2 + s_2^2) = [f^2 + c_s^2(\mathbf{k}^2 + 1/4\mathcal{H}^2)], \quad (39)$$

$$b = s_1^2 s_2^2 = c_s^2[k_H^2 N_B^2 + f^2(m^2 + 1/4\mathcal{H}^2)], \quad (40)$$

the 2 roots of Eq. (38) (corresponding to the GW and AW, respectively) are

$$s_1^2 = -\omega_1^2 = -\frac{a}{2}\left[1 - \sqrt{1 - 4b/a^2}\right], \quad (41)$$

$$s_2^2 = -\omega_2^2 = -\frac{a}{2}\left[1 + \sqrt{1 - 4b/a^2}\right]. \quad (42)$$

The intrinsic and ground-based wave frequencies are therefore $\omega_{Ir} = -is$ and $\omega_r = -is_r$, respectively. Using these definitions, Eq. (31) becomes

$$\omega_{Ir} = \omega_r - (k\overline{U} + l\overline{V}). \quad (43)$$

This is the familiar expression for the Doppler shifting of a wave's frequency in a mean wind. The intrinsic GW frequency is $\omega_{\text{GW}} = \omega_1 = -is_1$, while the intrinsic AW frequency is $\omega_{\text{AW}} = \omega_2 = -is_2$. Eq. (38) then becomes the well-known, non-dissipative, acoustic-gravity wave dispersion relation first derived by Hines (1960) (see also Eq.(22) in FA03).

$$\omega_{Ir}^4 - [f^2 + c_s^2(\mathbf{k}^2 + 1/4\mathcal{H}^2)]\omega_{Ir}^2 + c_s^2[k_H^2 N_B^2 + f^2(m^2 + 1/4\mathcal{H}^2)] = 0. \quad (44)$$

For a GW which propagates much more slowly than the sound speed ($\omega_{\text{GW}}/\sqrt{\mathbf{k}^2 + 1/4\mathcal{H}^2} \ll c_s$), the usual f-plane anelastic GW dispersion relation is obtained from Eq. (41) for $4b/a^2 \ll 1$: $s_1^2 \simeq -b/a$, or

$${\omega_{\text{GW}}}^2 = \frac{k_H^2 N_B^2 + f^2(m^2 + 1/4\mathcal{H}^2)}{\mathbf{k}^2 + 1/4\mathcal{H}^2}. \quad (45)$$

The acoustic cut-off frequency is obtained from Eqs. (39) and (42) for $f = \mathbf{k}^2 = 0$ and $4b/a^2 \ll 1$: $\omega_a = \sqrt{a} = c_s/2\mathcal{H}$. Since $c_s^2 = \gamma g\mathcal{H}$, $\omega_a = \gamma g/2c_s = (\gamma/2\sqrt{\gamma-1})N_B$. For $\gamma = 1.4$ and $N_B = 0.02$ rad/s (buoyancy period of $\tau_b = 2\pi/N_B = 5.2$ min), the acoustic cut-off period is $2\pi/\omega_a = 4.7$ min.

The terms in the first square bracket in Eq. (32) incorporate only the initial conditions, while the terms in the second square bracket incorporate only the interval body forcings and heatings. The compressible f-plane polarization relations are given in Sec. 6.

3. Wave and Mean Responses to Initial Conditions

We now derive the initial value solutions. We set $\mathcal{L}_{\mathcal{F}} = 0$, and take the inverse Laplace transform of Eq. (32). Here, we use the fact that the inverse transform of $\mathcal{L}(\widetilde{\psi})$ is

$$\widetilde{\psi}(t) = \int_0^\infty \mathrm{e}^{s_r t}\mathcal{L}(\widetilde{\psi})ds_r = \mathrm{e}^{i(k\overline{U}+l\overline{V})t}\int_0^\infty \mathrm{e}^{st}\mathcal{L}(\widetilde{\psi})ds, \tag{46}$$

where we have used Eq. (31), since $ds_r = ds$. We then solve Eqs.(19)-(23) with $\mathcal{F} = 0$. The compressible initial value solutions (denoted by the subscripts "IV") are then

$$\begin{aligned}\widetilde{\xi}_{IV}(t) &= \mathrm{e}^{i(k\overline{U}+l\overline{V})t}\left[\frac{i}{f^2-\omega_1^2}\left\{\left(\frac{lfE}{\omega_1} - k\omega_1 F\right)\sin\omega_1 t + (kE + lfF)\cos\omega_1 t\right\}\right.\\ &\left. + \frac{i}{f^2-\omega_2^2}\left\{\left(\frac{lfG}{\omega_2} - k\omega_2 H\right)\sin\omega_2 t + (kG + lfH)\cos\omega_2 t\right\} + \frac{ilI}{f}\right] \end{aligned}\tag{47}$$

$$\begin{aligned}\widetilde{\sigma}_{IV}(t) &= \mathrm{e}^{i(k\overline{U}+l\overline{V})t}\left[\frac{i}{f^2-\omega_1^2}\left\{-\left(\frac{kfE}{\omega_1} + l\omega_1 F\right)\sin\omega_1 t + (lE - kfF)\cos\omega_1 t\right\}\right.\\ &\left. + \frac{i}{f^2-\omega_2^2}\left\{-\left(\frac{kfG}{\omega_2} + l\omega_2 H\right)\sin\omega_2 t + (lG - kfH)\cos\omega_2 t\right\} - \frac{ikI}{f}\right] \end{aligned}\tag{48}$$

$$\begin{aligned}\widetilde{\eta}_{IV}(t) &= \mathrm{e}^{i(k\overline{U}+l\overline{V})t}\left(im_s - \frac{1}{\gamma\mathcal{H}}\right)\left\{\frac{1}{N_B^2-\omega_1^2}\left(E\cos\omega_1 t - F\omega_1\sin\omega_1 t\right)\right.\\ &\left. + \frac{1}{N_B^2-\omega_2^2}\left(G\cos\omega_2 t - H\omega_2\sin\omega_2 t\right)\right\} \end{aligned}\tag{49}$$

$$\begin{aligned}\widetilde{\phi}_{IV}(t) &= \mathrm{e}^{i(k\overline{U}+l\overline{V})t}\left[\frac{(-\omega_1^2 + im_s(\gamma-1)g)}{c_s^2(N_B^2-\omega_1^2)}\left(\frac{E}{\omega_1}\sin\omega_1 t + F\cos\omega_1 t\right)\right.\\ &\left. + \frac{(-\omega_2^2 + im_s(\gamma-1)g)}{c_s^2(N_B^2-\omega_2^2)}\left(\frac{G}{\omega_2}\sin\omega_2 t + H\cos\omega_2 t\right) + \frac{im_s}{g}I\right] \end{aligned}\tag{50}$$

$$\widetilde{\psi}_{IV}(t) = \mathrm{e}^{i(k\overline{U}+l\overline{V})t}\left[\frac{E}{\omega_1}\sin\omega_1 t + F\cos\omega_1 t + \frac{G}{\omega_2}\sin\omega_2 t + H\cos\omega_2 t + I\right], \tag{51}$$

where

$$\epsilon = im_s g\widetilde{\psi}(0) + g\widetilde{\phi}(0)\delta \tag{52}$$

$$E = \frac{1}{s_2^2 - s_1^2}\left[\delta(s_1^2+f^2)\widetilde{\eta}(0) - ic_s^2\alpha(s_1^2+N_B^2)\right] \tag{53}$$

$$F = -\frac{1}{s_1^2(s_2^2-s_1^2)}\left[(\widetilde{\psi}(0)s_1^2+\epsilon)(s_1^2+f^2) + ic_s^2\beta f(s_1^2+N_B^2)\right] \tag{54}$$

$$G = \frac{1}{s_2^2 - s_1^2}\left[-\delta(s_2^2+f^2)\widetilde{\eta}(0) + ic_s^2\alpha(s_2^2+N_B^2)\right] \tag{55}$$

$$H = \frac{1}{s_2^2(s_2^2-s_1^2)}\left[(\widetilde{\psi}(0)s_2^2+\epsilon)(s_2^2+f^2) + ic_s^2\beta f(s_2^2+N_B^2)\right] \tag{56}$$

$$I = \frac{f}{s_1^2 s_2^2}(f\epsilon + ic_s^2\beta N_B^2). \tag{57}$$

Special case solutions are given in Appendix A. Note that the GW and AW solutions are separate and distinct; the GW terms are proportional to $\sin\omega_1 t$ and $\cos\omega_1 t$, while the AW terms are proportional to $\sin\omega_2 t$ and $\cos\omega_2 t$. We compare our solutions in the Boussinesq limit (i.e., $\mathcal{H}\to\infty$ and $c_s^2\to\infty$) with the Boussinesq initial value solutions derived in VF01. First, we convert the density to potential temperature θ. Combining Eq. (6) with the ideal gas law and linearizing, we get

$$\frac{\rho'}{\overline{\rho}} = \frac{1}{c_s^2}\frac{p'}{\overline{\rho}} - \frac{\theta'}{\overline{\theta}}. \tag{58}$$

Neglecting the pressure perturbation in the Boussinesq approximation (since it is proportional to $1/c_s^2$), we find that

$$\theta'/\overline{\theta} \simeq -\rho'/\overline{\rho}. \tag{59}$$

It is then easy to show that Eqs. (47)-(51) reduce to Eqs. (B.1)-(B.5) in VF01.

4. Wave and mean responses to interval forcings/heatings

The function $\mathcal{F}(t)$ is the temporal portion of the forcing/heating, which turns on and off smoothly for a finite interval in time in order to represent the temporal evolution of realistic body forcings/heatings. In order to take the inverse Laplace transform of Eq. (32), $\mathcal{F}(t)$ must be a reasonably simple analytic function. Here, we choose $\mathcal{F}$ to be the same function as in VF01:

$$\mathcal{F}(t) = \frac{1}{\chi}\begin{cases} (1-\cos\hat{a}t) & \text{for } 0 \le t \le \chi \\ 0 & \text{for } t \le 0 \text{ and } t \ge \chi. \end{cases} \tag{60}$$

The interval forcing/heating lasts from $t = 0$ to χ, and has a frequency, $\hat{a}$, of

$$\hat{a} \equiv 2\pi n/\chi. \tag{61}$$

The number of cycles within the total duration χ is $n = 1, 2, 3, ...$, a positive integer. If $n = 1$, then only a single forcing/heating occurs over χ. An impulsive forcing is a special case of this more general forcing, and is obtained by setting $n = 1$ and $\chi \to 0$. The Laplace transform of Eq. (60) is

$$\mathcal{L}_{\mathcal{F}} = \frac{1}{\chi}(1-\mathrm{e}^{-s_r\chi})\frac{\hat{a}^2}{s_r(s_r^2+\hat{a}^2)}. \tag{62}$$

For simplicity, we set the background mean winds to zero here. Then $s = s_r$ from Eq. (31). We plug Eq. (62) into Eq. (32), and set all terms in the first square bracket equal to zero (i.e., $\widetilde{\xi}(0) = \widetilde{\sigma}(0) = \widetilde{\eta}(0) = \widetilde{\phi}(0) = \widetilde{\psi}(0) = 0$). We then take the inverse Laplace transform of Eq. (32) to determine $\widetilde{\psi}$. Finally, we solve Eqs.(19)-(23) to determine the other variables. The forced/heated solutions (denoted by the subscripts "FH") during the forcing/heating (i.e., when $0 \le t \le \chi$) are

$$\begin{aligned}
\widetilde{\xi}_{FH}(t) \;=\; & \frac{1}{\chi}\Bigg[i\hat{a}^2 l\frac{K}{f}t + \frac{(\cos\hat{a}t-1)}{f^2-\hat{a}^2}\left\{i\hat{a}^2(kM+lfN) - f\widetilde{F_{ys}}\right\} \\
& + \frac{\hat{a}\sin\hat{a}t}{f^2-\hat{a}^2}\left\{i(-k\hat{a}^2N+lfM)+\widetilde{F_{xs}}\right\} \\
& + i\hat{a}^2\Bigg\{\frac{1}{f^2-\omega_1^2}\left[(kO+lfP)(\cos\omega_1 t-1)+(-k\omega_1 P+\frac{lfO}{\omega_1})\sin\omega_1 t\right] \\
& + \frac{1}{f^2-\omega_2^2}\left[(kQ+lfR)(\cos\omega_2 t-1)+(-k\omega_2 R+\frac{lfQ}{\omega_2})\sin\omega_2 t\right]\Bigg\}\Bigg]
\end{aligned} \tag{63}$$

$$\widetilde{\sigma}_{FH}(t) \;=\; \frac{1}{\chi}\Bigg[-i\hat{a}^2 k\frac{K}{f}t + \frac{(\cos\hat{a}t-1)}{f^2-\hat{a}^2}\left\{i\hat{a}^2(lM-kfN)+f\widetilde{F_{xs}}\right\}$$

$$
\begin{aligned}
&+\frac{\hat{a}\sin\hat{a}t}{f^2-\hat{a}^2}\left\{-i(l\hat{a}^2N+kfM)+\widetilde{F_{ys}}\right\}\\
&+i\hat{a}^2\left\{\frac{1}{f^2-\omega_1^2}\left[(lO-kfP)(\cos\omega_1 t-1)-(l\omega_1 P+\frac{kfO}{\omega_1})\sin\omega_1 t\right]\right.\\
&\left.+\frac{1}{f^2-\omega_2^2}\left[(lQ-kfR)(\cos\omega_2 t-1)-(l\omega_2 R+\frac{kfQ}{\omega_2})\sin\omega_2 t\right]\right\}\Bigg]
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
\widetilde{\eta}_{FH}(t) &= \frac{1}{\chi\gamma\mathcal{H}}\left[\frac{1}{N_B^2-\hat{a}^2}\left\{\left[\hat{a}^2(i\gamma\mathcal{H}m_s-1)M-\widetilde{J_s}\right](\cos\hat{a}t-1)\right.\right.\\
&\left.+\hat{a}\left[-\hat{a}^2(i\gamma\mathcal{H}m_s-1)N+\gamma\mathcal{H}\widetilde{F_{zs}}\right]\sin\hat{a}t\right\}\\
&+\hat{a}^2(i\gamma\mathcal{H}m_s-1)\left\{\frac{1}{N_B^2-\omega_1^2}\left[O(\cos\omega_1 t-1)-P\omega_1\sin\omega_1 t\right]\right.\\
&\left.\left.+\frac{1}{N_B^2-\omega_2^2}\left[Q(\cos\omega_2 t-1)-R\omega_2\sin\omega_2 t\right]\right\}\right]
\end{aligned}
\tag{65}
$$

$$
\begin{aligned}
\widetilde{\phi}_{FH}(t) &= \frac{1}{\chi c_s^2}\left[\frac{im_s g(\gamma-1)\hat{a}^2}{N_B^2}Kt-\frac{\hat{a}\sin\hat{a}t}{N_B^2-\hat{a}^2}\left\{M(\hat{a}^2-im_s g(\gamma-1))+\widetilde{J_s}\right\}\right.\\
&-\frac{(\cos\hat{a}t-1)}{N_B^2-\hat{a}^2}\left\{\hat{a}^2N(\hat{a}^2-im_s g(\gamma-1))+g(\gamma-1)\widetilde{F_{zs}}\right\}\\
&+\hat{a}^2\left\{\left(\frac{im_s g(\gamma-1)-\omega_1^2}{N_B^2-\omega_1^2}\right)\left[\frac{O}{\omega_1}\sin\omega_1 t+P(\cos\omega_1 t-1)\right]\right.\\
&\left.\left.+\left(\frac{im_s g(\gamma-1)-\omega_2^2}{N_B^2-\omega_2^2}\right)\left[\frac{Q}{\omega_2}\sin\omega_2 t+R(\cos\omega_2 t-1)\right]\right\}\right]
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
\widetilde{\psi}_{FH}(t) &= \frac{\hat{a}^2}{\chi}\left[Kt+\frac{M}{\hat{a}}\sin\hat{a}t+N(\cos\hat{a}t-1)+\frac{O}{\omega_1}\sin\omega_1 t+P(\cos\omega_1 t-1)\right.\\
&\left.+\frac{Q}{\omega_2}\sin\omega_2 t+R(\cos\omega_2 t-1)\right].
\end{aligned}
\tag{67}
$$

The forced/heated solutions after the forcing/heating is finished (i.e., when $t \geq \chi$) are

$$
\begin{aligned}
\widetilde{\xi}_{FH}(t) &= \frac{i\hat{a}^2}{\chi}\left[l\chi\frac{K}{f}+\frac{1}{f^2-\omega_1^2}\left\{(kO+lfP)\mathcal{C}(\omega_1)+(-k\omega_1 P+\frac{lfO}{\omega_1})\mathcal{S}(\omega_1)\right\}\right.\\
&\left.+\frac{1}{f^2-\omega_2^2}\left\{(kQ+lfR)\mathcal{C}(\omega_2)+(-k\omega_2 R+\frac{lfQ}{\omega_2})\mathcal{S}(\omega_2)\right\}\right]
\end{aligned}
\tag{68}
$$

$$
\begin{aligned}
\widetilde{\sigma}_{FH}(t) &= \frac{i\hat{a}^2}{\chi}\left[-k\chi\frac{K}{f}+\frac{1}{f^2-\omega_1^2}\left\{(lO-kfP)\mathcal{C}(\omega_1)-(l\omega_1 P+\frac{kfO}{\omega_1})\mathcal{S}(\omega_1)\right\}\right.\\
&\left.+\frac{1}{f^2-\omega_2^2}\left\{(lQ-kfR)\mathcal{C}(\omega_2)-(l\omega_2 R+\frac{kfQ}{\omega_2})\mathcal{S}(\omega_2)\right\}\right]
\end{aligned}
\tag{69}
$$

$$
\widetilde{\eta}_{FH}(t) = \frac{\hat{a}^2(i\gamma\mathcal{H}m_s-1)}{\chi\gamma\mathcal{H}}\left[\frac{1}{N_B^2-\omega_1^2}\left[O\mathcal{C}(\omega_1)-P\omega_1\mathcal{S}(\omega_1)\right]\right.
$$

$$+\frac{1}{N_B^2-\omega_2^2}\left[Q\mathcal{C}(\omega_2)-R\omega_2\mathcal{S}(\omega_2)\right]\Bigg] \tag{70}$$

$$\widetilde{\phi}_{FH}(t) = \frac{1}{\chi c_s^2}\left[\frac{im_sg(\gamma-1)\hat{a}^2}{N_B^2}K\chi+\hat{a}^2\left\{\left(\frac{im_sg(\gamma-1)-\omega_1^2}{N_B^2-\omega_1^2}\right)\left[\frac{O}{\omega_1}\mathcal{S}(\omega_1)+P\mathcal{C}(\omega_1)\right]\right.\right.$$

$$\left.\left.+\left(\frac{im_sg(\gamma-1)-\omega_2^2}{N_B^2-\omega_2^2}\right)\left[\frac{Q}{\omega_2}\mathcal{S}(\omega_2)+R\mathcal{C}(\omega_2)\right]\right\}\right] \tag{71}$$

$$\widetilde{\psi}_{FH}(t) = \frac{\hat{a}^2}{\chi}\left[K\chi+\frac{O}{\omega_1}\mathcal{S}(\omega_1)+P\mathcal{C}(\omega_1)+\frac{Q}{\omega_2}\mathcal{S}(\omega_2)+R\mathcal{C}(\omega_2)\right]. \tag{72}$$

Here, we have defined

$$\mathcal{S}(\omega) \equiv \sin\omega t-\sin\omega(t-\chi) \tag{73}$$

$$\mathcal{C}(\omega) \equiv \cos\omega t-\cos\omega(t-\chi) \tag{74}$$

$$K = \frac{f}{\hat{a}^2s_1^2s_2^2}\left(ic_s^2B_{\mathrm{F}}N_B^2+im_sgf\widetilde{J_s}\right) \tag{75}$$

$$\tag{76}$$

$$M = \frac{1}{\hat{a}^2(\hat{a}^2+s_1^2)(\hat{a}^2+s_2^2)}\left(-ic_s^2fB_{\mathrm{F}}(N_B^2-\hat{a}^2)+(\hat{a}^2-f^2)(im_sg-\hat{a}^2)\widetilde{J_s}\right) \tag{77}$$

$$N = \frac{1}{\hat{a}^2(\hat{a}^2+s_1^2)(\hat{a}^2+s_2^2)}\left(-ic_s^2A_{\mathrm{F}}(N_B^2-\hat{a}^2)-\delta(\hat{a}^2-f^2)\widetilde{F_{zs}}\right) \tag{78}$$

$$O = \frac{1}{s_1^2(s_2^2-s_1^2)(\hat{a}^2+s_1^2)}\left(-ic_s^2fB_{\mathrm{F}}(N_B^2+s_1^2)-(im_sg+s_1^2)(f^2+s_1^2)\widetilde{J_s}\right) \tag{79}$$

$$P = \frac{1}{s_1^2(s_2^2-s_1^2)(\hat{a}^2+s_1^2)}\left(-ic_s^2A_{\mathrm{F}}(N_B^2+s_1^2)+\delta(f^2+s_1^2)\widetilde{F_{zs}}\right) \tag{80}$$

$$Q = \frac{1}{s_2^2(s_2^2-s_1^2)(\hat{a}^2+s_2^2)}\left(ic_s^2fB_{\mathrm{F}}(N_B^2+s_2^2)+(im_sg+s_2^2)(f^2+s_2^2)\widetilde{J_s}\right) \tag{81}$$

$$R = \frac{1}{s_2^2(s_2^2-s_1^2)(\hat{a}^2+s_2^2)}\left(ic_s^2A_{\mathrm{F}}(N_B^2+s_2^2)-\delta(f^2+s_2^2)\widetilde{F_{zs}}\right). \tag{82}$$

The special case solutions are given in Appendix B. It can be shown that these forced/heated solutions equal the impulsive solutions when $\chi \to 0$ and $\overline{U}=\overline{V}=0$. As with the initial solutions, the GW and AW terms are separate and distinct; the GW terms are proportional to $\sin\omega_1 t$ and $\cos\omega_1 t$, while the AW terms are proportional to $\sin\omega_2 t$ and $\cos\omega_2 t$. Although these solutions are quite complicated, they are easily solvable analytically in specific limits (see Sec. 5) and numerically (see Sec. 6-8). Eqs. (63)-(72), (28) and (16) contain the GW and AW spectral solutions in (k, l, m) space. Taking the inverse Fourier transform yields the compressible solutions in real space.

5. Forced/Heated Compressible Solutions for $f = 0$

In this section, we calculate the compressible solutions for $f = 0$ after the forcings/heatings are finished. This is applicable for medium and high frequency waves (such that the Coriolis force can be neglected). We show that the compressible mean and GW solutions equal the Boussinesq mean and GW solutions in the limit that $\mathcal{H} \to \infty$ and $c_s^2 \to \infty$, as they should.

1. Zonal body force

The post-forcing ($t \geq \chi$) Boussinesq GW zonal and vertical velocity and potential temperature perturbations that arise from a zonal body force with $f = 0$ are (VF01):

$$\begin{aligned}
\widetilde{u'} &= \frac{l^2}{k_H^2}\widetilde{F_x} + \left\{\frac{k^2}{k_H^2}\left(1 - \frac{\omega_1^2}{N_B^2}\right)\frac{\hat{a}^2}{\chi\omega_1(\hat{a}^2 - \omega_1^2)}\mathcal{S}(\omega_1)\right\}\widetilde{F_x}, && (83)\\
\widetilde{w'} &= -\frac{km}{k_H^2}\frac{\omega_1^2}{N_B^2}\frac{\hat{a}^2}{\chi\omega_1(\hat{a}^2 - \omega_1^2)}\widetilde{F_x}\mathcal{S}(\omega_1), && (84)\\
\widetilde{\theta'}/\overline{\theta} &= -\frac{km\omega_1}{gk_H^2}\frac{\hat{a}^2}{\chi\omega_1(\hat{a}^2 - \omega_1^2)}\widetilde{F_x}\mathcal{C}(\omega_1). && (85)
\end{aligned}$$

Note that the zonal velocity perturbation is composed of mean and GW components. The post-forcing ($t \geq \chi$) compressible zonal and vertical velocity and density perturbations from a zonal body force with $f = 0$ (from Eqs. (68), (70)-(71), (16)) are

$$\begin{aligned}
\left(\widetilde{\mathrm{e}^{-z/2\mathcal{H}}u'}\right) &= \frac{l^2}{k_H^2}\left(\widetilde{\mathrm{e}^{-z/2\mathcal{H}}F_x}\right) + \frac{k^2}{N_B^2 k_H^2}\left[\frac{1}{1 - \omega_1^2/\omega_2^2}\right]\left(\widetilde{\mathrm{e}^{-z/2\mathcal{H}}F_x}\right)\\
&\quad \left\{\left(N_B^2 - \omega_1^2\right)\frac{\hat{a}^2\mathcal{S}(\omega_1)}{\chi\omega_1(\hat{a}^2 - \omega_1^2)} - \frac{\omega_1^2}{\omega_2^2}\left(N_B^2 - \omega_2^2\right)\frac{\hat{a}^2\mathcal{S}(\omega_2)}{\chi\omega_2(\hat{a}^2 - \omega_2^2)}\right\}, && (86)\\
\left(\widetilde{\mathrm{e}^{-z/2\mathcal{H}}w'}\right) &= \frac{km\omega_1^2}{k_H^2 N_B^2}\left[\frac{1 + i\left(\frac{2-\gamma}{2\gamma\mathcal{H}m}\right)}{1 - \omega_1^2/\omega_2^2}\right]\left(\widetilde{\mathrm{e}^{-z/2\mathcal{H}}F_x}\right)\\
&\quad \left\{-\frac{\hat{a}^2\mathcal{S}(\omega_1)}{\chi\omega_1(\hat{a}^2 - \omega_1^2)} + \frac{\hat{a}^2\mathcal{S}(\omega_2)}{\chi\omega_2(\hat{a}^2 - \omega_2^2)}\right\}, && (87)
\end{aligned}$$

$$(\mathrm{e}^{-\widetilde{z/2\mathcal{H}}}\rho'/\overline{\rho}) = \frac{km\omega_1^2}{gk_H^2}\left[\frac{1}{1-\omega_1^2/\omega_2^2}\right]\left(\mathrm{e}^{-\widetilde{z/2\mathcal{H}}}F_x\right)\frac{\hat{a}^2}{\chi}$$
$$\left\{\frac{\left[1+\frac{i}{\gamma m\mathcal{H}}\left(\frac{\omega_1^2}{N_B^2}-\frac{\gamma}{2}\right)\right]\mathcal{C}(\omega_1)}{\omega_1^2(\hat{a}^2-\omega_1^2)} - \frac{\left[1+\frac{i}{\gamma m\mathcal{H}}\left(\frac{\omega_2^2}{N_B^2}-\frac{\gamma}{2}\right)\right]\mathcal{C}(\omega_2)}{\omega_2^2(\hat{a}^2-\omega_2^2)}\right\} \quad (88)$$

Here, each widetilde " ~ " encompasses all factors within the parenthesis. Those terms proportional to $\mathcal{S}(\omega_1)$ and $\mathcal{C}(\omega_1)$ are the GW solutions, while those terms proportional to $\mathcal{S}(\omega_2)$ and $\mathcal{C}(\omega_2)$ are the AW solutions. There are several differences between the compressible and Boussinesq GW solutions. First, the zonal body force and the GW compressible solutions are multiplied by $\exp(-z/2\mathcal{H})$ before taking their Fourier transforms. Therefore, if the depth of the body force is much greater than $\mathcal{H}$, the compressible solutions are expected to differ significantly from the Boussinesq solutions. Second, the compressible GW solutions include the extra terms in square brackets in Eqs. (86)-(88). The numerator of the [ ] term in Eq. (87) equals one if $\mathcal{H}m \gg 1$ (i.e., if $\lambda_z \ll 2\pi\mathcal{H}$), and the denominators of the [ ] terms in Eqs. (86)-(88) equal one if $\omega_1^2/\omega_2^2 \ll 1$. From Eqs. (41)-(42), this later expression is satisfied if $4b/a^2 \ll 1$ or $2N_B/c_s \ll (\mathbf{k}^2+1/4\mathcal{H}^2)/k_H$. Setting $\omega_1^2 = {k_H}^2 N_B^2/(\mathbf{k}^2+1/4\mathcal{H}^2)$, this condition becomes $c_H \ll c_s/2$, where the horizontal phase speed is $c_H = \omega_1/k_H$.

We now compare Eqs. (86)-(88) with Eqs. (83)-(85). We see that the compressible mean and GW solutions simplify to the Boussinesq solutions for $\mathcal{H}\to\infty$ and $c_s^2\to\infty$, since $\omega_1^2/\omega_2^2 \simeq b/a^2 \propto c_s^{-2}$. Here, we have used the Boussinesq approximation that $\rho'/\overline{\rho} \simeq -\theta'/\overline{\theta}$ from Eq. (59).

Fig. 1 shows contours of ω_1^2/ω_2^2 as a function of the horizontal wavelength $\lambda_H = 2\pi/k_H$ and the vertical wavelength $\lambda_z = 2\pi/m$. Here, the buoyancy period is $\tau_b = 5.2$ min. We see that $\omega_1^2/\omega_2^2 \le 0.25$, even for very large c_H. Therefore, there are likely only small changes in the GW compressible amplitudes due to the factor $[1-\omega_1^2/\omega_2^2]$ in the denominators of Eqs. (86)-(87). The right y axis shows the corresponding values of

$m\mathcal{H}$. For vertical wavelengths of importance in the thermosphere, we see that $m\mathcal{H}$ is of order or smaller than 1. Therefore, we expect the GW velocity perturbations to be significantly affected by compressibility for $\lambda_z \gtrsim \pi\mathcal{H}$ as compared to the Boussinesq solutions. This suggests that the compressible amplitudes for GWs with $\lambda_z \gtrsim 30$ km may depart significantly from the Boussinesq amplitudes.

Finally, Eq. (87) shows that the AW and GW vertical velocity amplitudes are comparable when the forcings/heatings are essentially impulsive. Essentially impulsive forcings/heatings have $\chi \ll \tau_c$, where τ_c is the characteristic time found by plugging the spatial scales of the forcing/heating into the wave dispersion relation (VF01). This is because

$$\mathcal{S}(\omega) \simeq \omega\chi \cos\omega t, \tag{89}$$

$\hat{a}^2 \gg \omega_1^2$, and $\hat{a}^2 \gg \omega_2^2$ for impulsive forcings. However, the AW zonal velocity amplitudes are reduced by $\simeq \omega_1^2/N_B^2$ from the GW amplitudes, even for essentially impulsive forcings. For forcings with $n = 1$ and $\chi \gg 5$ min (typical of horizontal body forces created by geostrophic adjustment of the jet stream), the AW zonal and vertical and density perturbation amplitudes are further reduced by

$$(2\pi)^{-1}\hat{a}^3/\omega_2^3, \tag{90}$$

because $2\pi/\omega_2$ is smaller than 5 min. The GW amplitudes are only significantly reduced for GWs with periods $\tau_{Ir} < \chi/2$; GWs with periods $> \chi/2$ are still excited with significant amplitudes. Therefore, the AW source amplitudes are much smaller than the GW source amplitudes for horizontal forcings with $\chi \gg 5$ min. We explore the GW and AW solutions for several important examples in Vadas (2010).

2. Vertical body force

The post-forcing ($t \geq \chi$) GW Boussinesq zonal and vertical velocity and potential temperature perturbations resulting from a vertical body force with $f = 0$ are (VF01):

$$\widetilde{u'} = -\frac{km}{k_H^2}\frac{\omega_1^2}{N_B^2}\frac{\hat{a}^2}{\chi\omega_1(\hat{a}^2 - \omega_1^2)}\widetilde{F_z}\mathcal{S}(\omega_1), \tag{91}$$

$$\widetilde{w'} = \left(\frac{\omega_1}{N_B}\right)^2 \frac{\hat{a}^2}{\chi\omega_1(\hat{a}^2 - \omega_1^2)}\widetilde{F_z}\mathcal{S}(\omega_1), \tag{92}$$

$$\widetilde{\theta'}/\overline{\theta} = \frac{\omega_1}{g}\frac{\hat{a}^2}{\chi\omega_1(\hat{a}^2 - \omega_1^2)}\widetilde{F_z}\mathcal{C}(\omega_1). \tag{93}$$

No mean winds are created from vertical body forces. The post-forcing ($t \geq \chi$) compressible zonal and vertical velocity and density perturbations resulting from a vertical body force with $f = 0$ (from Eqs. (68), (70)-(71), and (16)) are

$$\widetilde{(\mathrm{e}^{-z/2\mathcal{H}}u')} = \frac{km\omega_1^2}{k_H^2 N_B^2}\left[\frac{1 + i\left(\frac{\gamma-2}{2\gamma\mathcal{H}m}\right)}{1 - \omega_1^2/\omega_2^2}\right]\left(\widetilde{\mathrm{e}^{-z/2\mathcal{H}}F_z}\right) \left\{-\frac{\hat{a}^2\mathcal{S}(\omega_1)}{\chi\omega_1(\hat{a}^2 - \omega_1^2)} + \frac{\hat{a}^2\mathcal{S}(\omega_2)}{\chi\omega_2(\hat{a}^2 - \omega_2^2)}\right\}, \tag{94}$$

$$\widetilde{(\mathrm{e}^{-z/2\mathcal{H}}w')} = \frac{m^2\omega_1^2}{k_H^2 N_B^2}\left[\frac{1 + \left(\frac{\gamma-2}{2\gamma\mathcal{H}m}\right)^2}{1 - \omega_1^2/\omega_2^2}\right]\frac{\hat{a}^2}{\chi}\left(\widetilde{\mathrm{e}^{-z/2\mathcal{H}}F_z}\right) \left\{\frac{\omega_1\mathcal{S}(\omega_1)}{(N_B^2 - \omega_1^2)(\hat{a}^2 - \omega_1^2)} - \frac{\omega_2\mathcal{S}(\omega_2)}{(N_B^2 - \omega_2^2)(\hat{a}^2 - \omega_2^2)}\right\}, \tag{95}$$

$$\widetilde{(\mathrm{e}^{-z/2\mathcal{H}}\rho'/\overline{\rho})} = -\frac{m^2\omega_1^2}{gk_H^2}\left[\frac{1 + i\left(\frac{\gamma-2}{2\gamma m\mathcal{H}}\right)}{1 - \omega_1^2/\omega_2^2}\right]\frac{\hat{a}^2}{\chi}\left(\widetilde{\mathrm{e}^{-z/2\mathcal{H}}F_z}\right) \left\{\frac{\left[1 + \frac{i}{\gamma m\mathcal{H}}\left(\frac{\omega_1^2}{N_B^2} - \frac{\gamma}{2}\right)\right]\mathcal{C}(\omega_1)}{(N_B^2 - \omega_1^2)(\hat{a}^2 - \omega_1^2)} - \frac{\left[1 + \frac{i}{\gamma m\mathcal{H}}\left(\frac{\omega_2^2}{N_B^2} - \frac{\gamma}{2}\right)\right]\mathcal{C}(\omega_2)}{(N_B^2 - \omega_2^2)(\hat{a}^2 - \omega_2^2)}\right\} \tag{96}$$

As before, those terms proportional to $\mathcal{S}(\omega_1)$ and $\mathcal{C}(\omega_1)$ are the GW solutions, while those terms proportional to $\mathcal{S}(\omega_2)$ and $\mathcal{C}(\omega_2)$ are the AW solutions. Since

$$\frac{m^2\omega_1^2}{k_H^2(N_B^2 - \omega_1^2)} \simeq 1 \tag{97}$$

and $\omega_1^2 \ll \omega_2^2$ in the Boussinesq limit, we verify that the compressible solutions equal the Boussinesq solutions for $\mathcal{H} \to \infty$ and $c_s^2 \to \infty$. As with the zonal body force, the vertical body force and GW perturbations are multiplied by $\exp(-z/2\mathcal{H})$ prior to taking their Fourier transforms for the compressible solution. This factor is important for vertical body forces much deeper than $\mathcal{H}$. Although the $(1-\omega_1^2/\omega_2^2)$ factors are not expected to significantly affect the compressible GW amplitudes (see Sec. 5a), the factors in the numerators of the square brackets in Eqs. (94)-(96) are expected to significantly alter the GW compressible amplitudes for $\lambda_z \gtrsim 30$ km (see Fig. 1).

For essentially impulsive forcings (i.e., $\chi \ll \tau_c$), the GW and AW zonal velocity amplitudes are nearly the same. However, the vertical velocity amplitude of the AW can be much larger than that of the GW by N_B^2/ω_1^2. For forces with $\chi \gg 5$ min, the AW amplitudes are suppressed by $(2\pi)^{-1}\hat{a}^3/\omega_2^3$ from their impulsive values. Additionally, while the amplitude of those GWs with periods $\tau < \chi/2$ (and $n = 1$) are significantly reduced, the amplitudes those GWs with $\tau > \chi$ equal their impulsive amplitudes. We explore the GW and AW solutions for the overshooting of a deep convective plume in Vadas (2010).

3. Heating

If we define J_B to be the Boussinesq external heating (i.e., written as $J(\mathbf{x})$ on the right-hand-side of Eq. (2.4) in VF01), then using Eq. (8), J on the right-hand-side of Eq. (3) is related to J_B via

$$J \simeq \frac{\gamma\mathcal{H}}{\mathrm{r}} J_B. \tag{98}$$

The post-forcing ($t \geq \chi$) Boussinesq, GW zonal and vertical velocity and potential temperature perturbations from a heating are (VF01):

$$\widetilde{u'} = \frac{km\omega_1}{k_H^2 N_B^2} \frac{\hat{a}^2}{\chi\omega_1(\hat{a}^2-\omega_1^2)} \widetilde{J_B}\mathcal{C}(\omega_1), \tag{99}$$

$$\widetilde{w'} = -\frac{\omega_1}{N_B^2} \frac{\hat{a}^2}{\chi\omega_1(\hat{a}^2-\omega_1^2)} \widetilde{J_B}\mathcal{C}(\omega_1), \tag{100}$$

$$\widetilde{\theta'}/\overline{\theta} = \frac{1}{g} \frac{\hat{a}^2}{\chi\omega_1(\hat{a}^2-\omega_1^2)} \widetilde{J_B}\mathcal{S}(\omega_1). \tag{101}$$

For the same heating, the post-forcing ($t \geq \chi$) compressible, zonal and vertical velocity GW perturbations (from Eqs. (68), (70), and (16)) are

$$
\begin{aligned}
(\widetilde{\mathrm{e}^{-z/2\mathcal{H}}u'}) &= \frac{km\omega_1^2}{k_H^2 N_B^2}\left[\frac{1}{1-\omega_1^2/\omega_2^2}\right]\left(\frac{\mathrm{r}}{\gamma\mathcal{H}}\widetilde{\mathrm{e}^{-z/2\mathcal{H}}J}\right)\frac{\hat{a}^2}{\chi} \\
&\left\{\frac{\left[1+i\left(\frac{\omega_1^2}{gm}-\frac{1}{2m\mathcal{H}}\right)\right]\mathcal{C}(\omega_1)}{\omega_1^2(\hat{a}^2-\omega_1^2)} - \frac{\left[1+i\left(\frac{\omega_2^2}{gm}-\frac{1}{2m\mathcal{H}}\right)\right]\mathcal{C}(\omega_2)}{\omega_2^2(\hat{a}^2-\omega_2^2)}\right\} \qquad (102)
\end{aligned}
$$

$$
\begin{aligned}
(\widetilde{\mathrm{e}^{-z/2\mathcal{H}}w'}) &= -\frac{m^2\omega_1^2}{k_H^2 N_B^2}\left[\frac{1+i\left(\frac{2-\gamma}{2\gamma\mathcal{H}m}\right)}{1-\omega_1^2/\omega_2^2}\right]\left(\frac{\mathrm{r}}{\gamma\mathcal{H}}\widetilde{\mathrm{e}^{-z/2\mathcal{H}}J}\right)\frac{\hat{a}^2}{\chi} \\
&\left\{\frac{\left[1+i\left(\frac{\omega_1^2}{gm}-\frac{1}{2m\mathcal{H}}\right)\right]\mathcal{C}(\omega_1)}{(N_B^2-\omega_1^2)(\hat{a}^2-\omega_1^2)} - \frac{\left[1+i\left(\frac{\omega_2^2}{gm}-\frac{1}{2m\mathcal{H}}\right)\right]\mathcal{C}(\omega_2)}{(N_B^2-\omega_2^2)(\hat{a}^2-\omega_2^2)}\right\}, \qquad (103)
\end{aligned}
$$

$$
\begin{aligned}
(\widetilde{\mathrm{e}^{-z/2\mathcal{H}}\rho'/\overline{\rho}}) &= -\frac{m^2\omega_1^2}{gk_H^2}\left[\frac{1}{1-\omega_1^2/\omega_2^2}\right]\frac{\left(\widetilde{\mathrm{e}^{-z/2\mathcal{H}}J}\right)}{\gamma\mathcal{H}}\frac{\hat{a}^2}{\chi} \\
&\left\{\frac{\left[1+\frac{i}{m\gamma\mathcal{H}}\left(\frac{\omega_1^2}{N_B^2}-\frac{\gamma}{2}\right)\right]\left[1+i\left(\frac{\omega_1^2}{gm}-\frac{1}{2m\mathcal{H}}\right)\right]\mathcal{S}(\omega_1)}{\omega_1(N_B^2-\omega_1^2)(\hat{a}^2-\omega_1^2)}\right. \\
&\left. -\frac{\left[1+\frac{i}{m\gamma\mathcal{H}}\left(\frac{\omega_2^2}{N_B^2}-\frac{\gamma}{2}\right)\right]\left[1+i\left(\frac{\omega_2^2}{gm}-\frac{1}{2m\mathcal{H}}\right)\right]\mathcal{S}(\omega_2)}{\omega_2(N_B^2-\omega_2^2)(\hat{a}^2-\omega_2^2)}\right\}. \qquad (104)
\end{aligned}
$$

As before, those terms proportional to $\mathcal{S}(\omega_1)$ and $\mathcal{C}(\omega_1)$ are the GW solutions, while those terms proportional to $\mathcal{S}(\omega_2)$ and $\mathcal{C}(\omega_2)$ are the AW solutions. Using Eq. (98) and the fact that $m^2\omega_1^2 = k_H^2(N_B^2-\omega_1^2)$ in the Boussinesq limit, we verify that the compressible solutions equal the Boussinesq solutions for $\mathcal{H}\to\infty$ and $c_s^2\to\infty$. The heating and GW solutions are again multiplied by $\exp(-z/2\mathcal{H})$ prior to taking the Fourier transforms for the compressible solutions. This suggests that the compressible solutions will differ substantially from the Boussinesq solutions for heatings deeper than $\mathcal{H}$. Additionally, because the numerators of the GW terms in square brackets are proportional to $1/m\mathcal{H}$, it is expected that the compressible amplitudes will be different from the Boussinesq amplitudes for GWs with $\lambda_z \gtrsim 30$ km. We explore the GW and AW solutions for a simple auroral heating in Vadas (2010).

6. f-Plane Polarization Relations

In this section, we derive the compressible f-plane polarization relations for GWs and AWs. We show how these reduce to the Boussinesq polarization relations. We then examine the phase relationships between the GW perturbation velocities, temperatures, and densities for varying $\lambda_z/4\pi\mathcal{H}$.

1. Compressible polarization relations

We derive the exact, non-dissipative, compressible f-plane polarization relations via assuming GW and AW plane wave solutions of the form

$$\xi(x, y, z, t) = \mathrm{e}^{i(\omega_r t - kx - ly - mz)}\ \widetilde{\xi}(k, l, m). \tag{105}$$

This follows the same form as Eq. (18). We define the "hatted" quantities as:

$$\begin{aligned}
\hat{u} &= (\widetilde{\mathrm{e}^{-z/2\mathcal{H}} u'}) = \widetilde{\xi}, \qquad \hat{v} = (\widetilde{\mathrm{e}^{-z/2\mathcal{H}} v'}) = \widetilde{\sigma}, && (106)\\
\hat{w} &= (\widetilde{\mathrm{e}^{-z/2\mathcal{H}} w'}) = \widetilde{\eta}, \qquad \hat{\rho} = (\widetilde{\mathrm{e}^{-z/2\mathcal{H}} \rho'/\overline{\rho}}) = \widetilde{\phi}, && (107)\\
\hat{p} &= (\widetilde{\mathrm{e}^{-z/2\mathcal{H}} p'/\overline{\rho}}) = \widetilde{\psi}, \qquad \hat{T} = (\widetilde{\mathrm{e}^{-z/2\mathcal{H}} T'/\overline{T}}) = \widetilde{\zeta}, && (108)
\end{aligned}$$

where the widetilde " $\widetilde{\ }$ " encompasses all factors within each parenthesis. Using Eqs.(19)-(23) and (16), we obtain the compressible polarization relations:

$$\begin{aligned}
\hat{u} &= \frac{ik\omega_{Ir} + fl}{il\omega_{Ir} - fk}\ \hat{v}, && (109)\\
\hat{p} &= \frac{i(\omega_{Ir}^2 - f^2)}{ik\omega_{Ir} + fl}\ \hat{u} = \frac{i(\omega_{Ir}^2 - f^2)}{il\omega_{Ir} - fk}\ \hat{v}, && (110)\\
\hat{w} &= \frac{-\omega_{Ir}}{(N_B^2 - \omega_{Ir}^2)}\left(m - \frac{i}{2\mathcal{H}} + \frac{i}{\gamma\mathcal{H}}\right)\ \hat{p}, && (111)\\
\hat{\rho} &= \frac{[i(m - \frac{i}{2\mathcal{H}})N_B^2 - \frac{\omega_{Ir}^2}{\gamma\mathcal{H}}]}{g(N_B^2 - \omega_{Ir}^2)}\ \hat{p}, && (112)\\
\hat{T} &= \frac{-N_B^2(im - \frac{1}{2\mathcal{H}}) + \frac{\omega_{Ir}^2}{\gamma\mathcal{H}}(1-\gamma)}{g(N_B^2 - \omega_{Ir}^2)}\ \hat{p}, && (113)\\
\hat{w} &= \frac{-\omega_{Ir}(m - \frac{i}{2\mathcal{H}} + \frac{i}{\gamma\mathcal{H}})(\omega_{Ir}^2 - f^2)(k\omega_{Ir} + ifl)}{(N_B^2 - \omega_{Ir}^2)(k^2\omega_{Ir}^2 + f^2 l^2)}\ \hat{u}, && (114)
\end{aligned}$$

$$\hat{w} = \frac{-\omega_{Ir}(m - \frac{i}{2\mathcal{H}} + \frac{i}{\gamma\mathcal{H}})(\omega_{Ir}^2 - f^2)(l\omega_{Ir} - ifk)}{(N_B^2 - \omega_{Ir}^2)(l^2\omega_{Ir}^2 + f^2k^2)}\,\hat{v}, \tag{115}$$

$$\hat{w} = \frac{-g\omega_{Ir}(m - \frac{i}{2\mathcal{H}} + \frac{i}{\gamma\mathcal{H}})}{i(m - \frac{i}{2\mathcal{H}})N_B^2 - \frac{\omega_{Ir}^2}{\gamma\mathcal{H}}}\,\hat{\rho}, \tag{116}$$

$$\hat{T} = \frac{N_B^2(im - \frac{1}{2\mathcal{H}}) - \frac{\omega_{Ir}^2}{\gamma\mathcal{H}}(1-\gamma)}{g\omega_{Ir}(m - \frac{i}{2\mathcal{H}} + \frac{i}{\gamma\mathcal{H}})}\,\hat{w}. \tag{117}$$

Note that if the GWs and AWs are assumed to have the oscillatory form $e^{i(-\omega_r t + kx + ly + mz)}$ instead, then one must replace i by $-i$ in Eqs. (109)-(117) to obtain the corresponding compressible polarization relations. In the limit that $f = 0$, it is easy to show that Eqs. (114), (116), and (117) agree with the compressible, dissipative polarization relations derived in Vadas and Fritts (2005) (Eqs. (B1)-(B3) in that paper) in the limit that $\nu = 0$ (and for $i \to -i$). Note also that Eq. (110) agrees with Eq. (27) in FA03. However, Eq. (111) differs significantly from Eq. (28) in FA03; we conclude that Eq. (28) in FA03 has a sign error and neglects a term proportional to $1/\mathcal{H}$.

2. Boussinesq polarization relations

For GWs with $\lambda_z \ll 4\pi\mathcal{H}$ (or $\lambda_z \lesssim 30$ km in the lower atmosphere), Eqs. (109)-(110) are exactly the same. However, the rest of the compressible polarization relations reduce to the following Boussinesq polarization relations:

$$\hat{w} = \frac{-\omega_{Ir}m}{(N_B^2 - \omega_{Ir}^2)}\,\hat{p}, \tag{118}$$

$$\hat{\rho} = \frac{imN_B^2}{g(N_B^2 - \omega_{Ir}^2)}\,\hat{p}, \tag{119}$$

$$\hat{T} = \frac{-imN_B^2}{g(N_B^2 - \omega_{Ir}^2)}\,\hat{p}, \tag{120}$$

$$\hat{w} = \frac{-\omega_{Ir}m(\omega_{Ir}^2 - f^2)(k\omega_{Ir} + ifl)}{(N_B^2 - \omega_{Ir}^2)(k^2\omega_{Ir}^2 + f^2l^2)}\,\hat{u}, \tag{121}$$

$$\hat{w} = \frac{-\omega_{Ir}m(\omega_{Ir}^2 - f^2)(l\omega_{Ir} - ifk)}{(N_B^2 - \omega_{Ir}^2)(l^2\omega_{Ir}^2 + f^2k^2)}\,\hat{v}, \tag{122}$$

$$\hat{w} = \frac{ig\omega_{Ir}}{N_B^2}\,\hat{\rho}, \tag{123}$$

$$\hat{T} = \frac{iN_B^2}{g\omega_{Ir}}\,\hat{w}. \tag{124}$$

If the GWs have the oscillatory form $e^{i(-\omega_r t+kx+ly+mz)}$ instead, one must replace i by $-i$ in Eqs. (118)-(124) to obtain the corresponding polarization relations.

3. Phase relationships in the thermosphere for high-frequency GWs

We now examine the phase relationships of high-frequency GWs in the lower atmosphere. We therefore can assume that $f = 0$. From Eq. (109), $\hat{u}$ and $\hat{v}$ are always 0 or 180 degrees out of phase, regardless of $\lambda_z/4\pi\mathcal{H}$. This is not the case for the vertical velocity perturbation. For $\lambda_z \ll 4\pi\mathcal{H}$ and $f = 0$, Eq. (121) becomes

$$\hat{w} = \frac{-m}{k}\frac{\omega_{Ir}^2}{(N_B^2 - \omega_{Ir}^2)}\,\hat{u} \rightarrow_{l=0,m^2\gg k^2}\rightarrow \frac{-k}{m}\,\hat{u}. \tag{125}$$

The arrow denotes the familiar solution for eastward-propagating GWs ($l = 0$) with $m^2 \gg k^2$. In this case, if the GW is propagating upward (i.e., $m < 0$), then $\hat{w}$ is 0 (180) degrees out of phase with $\hat{u}$ if the GW is propagating eastward (westward). This is the familiar Boussinesq relationship. However, if $\lambda_z \gtrsim 4\pi\mathcal{H}$, the phase relationship from Eq. (114) is

$$\hat{w} = \frac{-m}{k}\left(1 + \frac{i(2-\gamma)}{2m\gamma\mathcal{H}}\right)\frac{\omega_{Ir}^2}{(N_B^2 - \omega_{Ir}^2)}\,\hat{u}. \tag{126}$$

Since $i = \exp(i\pi/2)$, $\hat{w}$ has a different phase relationship with $\hat{u}$ when $|\lambda_z/4\pi\mathcal{H}| \gtrsim 1$. Figure 2a shows these solutions for upward and eastward-propagating GWs in the lower atmosphere with $\lambda_x = 300$ km and for $\lambda_z/4\pi\mathcal{H} = -.11$, -1.1, and -3.4. Here, we choose $\mathcal{H} = 7$ km and $\gamma = 1.4$, characteristic of the lower atmosphere. Additionally, $f = 0$, $y = z = t = 0$, and the magnitude of $\hat{u}$ is chosen to be 15 m/s for all GWs. Here, we plot all $\hat{w}$ curves normalized to 25 m/s in order to better see their phase shifts with respect to $\hat{u}$. As expected, $\hat{w}$ is nearly exactly in phase with $\hat{u}$ for the GW with $\lambda_z/4\pi\mathcal{H} = -.11$, in good agreement with the Boussinesq solution. However, as $|\lambda_z/4\pi\mathcal{H}|$ increases to be larger than one, $\hat{w}$ becomes significantly phase-shifted from $\hat{u}$. Note that although the

$\hat{w}$ curves are phase-shifted, these phase shifts preserve the sinusoidal properties of $\hat{w}$. Figure 2b shows $\hat{u}$ and $11\hat{w}$ for the GW with $|\lambda_z| = 300$ km. For this GW, $\hat{w}$ lags $\hat{u}$ (in x) by -55^{o}. Because $\mathcal{H}$ is at least ~ 3 times larger in the thermosphere, significant phase shifts will occur for GWs in the thermosphere for correspondingly larger values of λ_z.

In the Boussinesq approximation, the temperature and density perturbations are $\pm 90^{\text{o}}$ out of phase with the horizontal velocity perturbations (see Eqs. (121), (123), and (124)). However, this is not true for GWs with very large λ_z (see Eqs. (114), (116), and (117)). Fig. 3a shows the temperature perturbations for varying $\lambda_z/4\pi\mathcal{H}$ for the same atmospheric conditions as in Fig. 2. While the magnitude of $\hat{T}$ does not vary significantly, its phase changes substantially with $\lambda_z/4\pi\mathcal{H}$. Although $\hat{T}$ leads $\hat{u}$ (in x) by 90^{o} for small $|\lambda_z|$, the phase of $\hat{T}$ decreases significantly as $|\lambda_z|$ increases. For $\lambda_z = -300$ km, $\hat{T}$ leads $\hat{u}$ by only $\sim 15 - 20^{\text{o}}$.

Fig. 3b shows the variation of $\rho'/\overline{\rho}$ as a function of $\lambda_z/4\pi\mathcal{H}$. Although the magnitude of $\hat{\rho}$ does not vary substantially, the phase depends significantly on $\lambda_z/4\pi\mathcal{H}$. While $\hat{\rho}$ lags $\hat{u}$ (in x) by 90^{o} for small $|\lambda_z|$, the phase of $\hat{\rho}$ increases significantly as $|\lambda_z|$ increases. For $\lambda_z = -300$ km, $\hat{\rho}$ lags $\hat{u}$ by only $\sim 15 - 20^{\text{o}}$. Fig. 3c and d shows $\hat{T}$ and $\hat{\rho}$ for $\lambda_z = -10$ and -300 km, respectively. As expected from the Boussinesq relations, $\hat{T}$ and $\hat{\rho}$ are 180^{o} out of phase when $|\lambda_z/4\pi/\mathcal{H}| \ll 1$. However, for $\lambda_z = -300$ km, $\hat{T}$ and $\hat{\rho}$ are nearly in phase ($\hat{\rho}$ lags $\hat{T}$ by only $\sim 40^{\text{o}}$).

7. Conclusions

In this paper, we derived the compressible f-plane solutions to linear horizontal and vertical body forcings and heatings. The solutions consist of mean, GW, and AW responses. We showed that the compressible amplitudes for those excited GWs with $\lambda_z \lesssim 2\pi\mathcal{H}$ are nearly identical to the Boussinesq amplitudes. However, for GWs with $\lambda_z \gtrsim 2\pi\mathcal{H}$, the solutions are expected to differ substantially. In the lower atmosphere, the compressible solutions are necessary for GWs with $\lambda_z \gtrsim 50$.

As with the Boussinesq solutions, there is a significant cut-off factor. For forcing frequencies of $\hat{a}$, the amplitudes of the AWs and GWs are with frequencies larger than $2\hat{a}$ are severely damped by $(2\pi)^{-1}\hat{a}^3/\omega^3$ from their impulsive values. However, the amplitudes of the AWs and GWs with frequencies smaller than $\hat{a}/2$ are not significantly changed from their impulsive values.

We found that the AW amplitudes are similar to the GW amplitudes for nearly impulsive forcing durations. However, because AWs have frequencies smaller than ~ 5 min, forcings/heatings with durations $\chi \gg 5$ min (and $n = 1$) excite AWs with very small amplitudes.

We also derived the exact, non-dissipative, compressible f-plane polarization relations for GWs and AWs. We found that the phase relationships between u', v', w', $T'/\overline{T}$, and $\rho'/\overline{\rho}$ are far more complicated than the Boussinesq relationships for waves with $\lambda_z \gtrsim \pi\mathcal{H}$. For example, for an eastward, upward-propagating GW with $\lambda_x = 300$ km and very large λ_z, u' and w' are 55° out of phase (instead of being 0° out of phase in the Boussinesq approximation), and $T'/\overline{T}$ and $\rho'/\overline{\rho}$ are only 50° out of phase (instead of being 180° out of phase in the Boussinesq approximation).

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APPENDIX A

Special Case of the Initial Condition Solutions

When $k = l = k_H = 0$, then $\omega_1 = f$ from Eq. (41). This causes the denominators of the GW terms in Eqs. (47) and (48) to equal zero. Therefore, we calculate $\widetilde{\xi}_{IV}(t)$ and $\widetilde{\sigma}_{IV}(t)$ directly from Eqs. (19)-(20) in this special case, since the horizontal velocity perturbations decouple from the pressure, vertical velocity, and density perturbations:

$$\widetilde{\xi}_{IV}(t) = \widetilde{\sigma}(0)\sin ft + \widetilde{\xi}(0)\cos ft, \tag{A.1}$$

$$\widetilde{\sigma}_{IV}(t) = -\widetilde{\xi}(0)\sin ft + \widetilde{\sigma}(0)\cos ft. \tag{A.2}$$

When $k = l = k_H = 0$ and $f = 0$, then $\omega_1 = 0$. This causes the denominators of F and I to equal zero (see Eqs. (54) and (57)). Instead, the solutions in this special case are

$$\widetilde{\eta}_{IV}(t) = \frac{im_s - 1/(\gamma\mathcal{H})}{{N_B}^2 - \omega_2^2}\left(G'\cos\omega_2 t - H'\omega_2\sin\omega_2 t\right) \tag{A.3}$$

$$\widetilde{\phi}_{IV}(t) = \frac{-\omega_2^2 + im_s(\gamma-1)g}{c_s^2({N_B}^2 - \omega_2^2)}\left(\frac{G'}{\omega_2}\sin\omega_2 t + H'\cos\omega_2 t\right) + \frac{im_s}{g}I' \tag{A.4}$$

$$\widetilde{\psi}_{IV}(t) = \left(\frac{G'}{\omega_2}\sin\omega_2 t + H'\cos\omega_2 t\right) + I', \tag{A.5}$$

where

$$G' = -\delta\widetilde{\eta}(0), \qquad H' = \widetilde{\psi}(0) + \epsilon/s_2^2, \qquad I' = -\epsilon/s_2^2. \tag{A.6}$$

APPENDIX B

Special Case of the Forced/Heated Solutions

When $k = l = k_H = 0$, then $\omega_1 = f$ from Eq. (41). This causes the denominators of Eqs. (63)-(64) and Eqs. (68)-(69) to equal zero. Therefore, we must calculate $\widetilde{\xi}_{FH}(t)$ and $\widetilde{\sigma}_{FH}(t)$ directly from Eqs. (19)-(20) in this special case, since the horizontal velocity perturbations decouple from the pressure, vertical velocity, and density perturbations.

During the forcing (i.e., when $t \le \chi$), the solutions are

$$\widetilde{\xi}_{FH}(t) = \frac{\hat{a}^2}{\chi(\hat{a}^2 - f^2)} \left[\widetilde{F_{xs}} \left\{ \frac{\sin ft}{f} - \frac{\sin \hat{a}t}{\hat{a}} \right\} + \frac{\widetilde{F_{ys}}}{f\hat{a}^2} \left\{ \hat{a}^2(1 - \cos ft) + f^2(\cos \hat{a}t - 1) \right\} \right] \tag{B.1}$$

$$\widetilde{\sigma}_{FH}(t) = \frac{\hat{a}^2}{\chi(\hat{a}^2 - f^2)} \left[\frac{\widetilde{F_{xs}}}{f\hat{a}^2} \left\{ \hat{a}^2(\cos ft - 1) - f^2(\cos \hat{a}t - 1) \right\} + \widetilde{F_{ys}} \left\{ \frac{\sin ft}{f} - \frac{\sin \hat{a}t}{\hat{a}} \right\} \right]. \tag{B.2}$$

After the forcing (i.e., when $t \ge \chi$), the solutions are

$$\widetilde{\xi}_{FH}(t) = \frac{\hat{a}^2}{\chi f(\hat{a}^2 - f^2)} \left[\widetilde{F_{xs}} \mathcal{S}(f) - \widetilde{F_{ys}} \mathcal{C}(f) \right] \tag{B.3}$$

$$\widetilde{\sigma}_{FH}(t) = \frac{\hat{a}^2}{\chi f(\hat{a}^2 - f^2)} \left[\widetilde{F_{xs}} \mathcal{C}(f) + \widetilde{F_{ys}} \mathcal{S}(f) \right]. \tag{B.4}$$

If $k = l = k_H = 0$ and $f = 0$, then the denominators of Eqs. (B.1)-(B.4) equal zero. For this special case, the horizontal velocity solutions are simple. During the forcing (i.e., when $t \le \chi$), the solutions are

$$\widetilde{\xi}_{FH}(t) = \frac{\widetilde{F_{xs}}}{\chi} \left[t - \frac{\sin \hat{a}t}{\hat{a}} \right] \tag{B.5}$$

$$\widetilde{\sigma}_{FH}(t) = \frac{\widetilde{F_{ys}}}{\chi} \left[t - \frac{\sin \hat{a}t}{\hat{a}} \right]. \tag{B.6}$$

After the forcing (i.e., when $t \ge \chi$), the solutions are

$$\widetilde{\xi}_{FH}(t) = \widetilde{F_{xs}} \tag{B.7}$$

$$\widetilde{\sigma}_{FH}(t) = \widetilde{F_{ys}}. \tag{B.8}$$

When $k = l = k_H = 0$ and $f = 0$, then $s_1 = 0$. This causes K, O, and P to be ∞ from Eqs. (75), (79), and (80). In this case, $\widetilde{\eta}_{FH}(t)$, $\widetilde{\phi}_{FH}(t)$ and $\widetilde{\psi}_{FH}(t)$ are given by Eqs. (65)-(67) during the forcing/heating, and by Eqs. (70)-(72) after the forcing/heating, but with the following replacements: $K \to K'$, $M \to M'$, $N \to N'$, $O \to O'$, $P \to P'$, $Q \to Q'$, and $R \to R'$, where

$$
\begin{aligned}
K' &= -\frac{i m_s g \widetilde{J_s}}{\hat{a}^2 s_2^2}, \qquad M' = \frac{(i m_s g - \hat{a}^2)\widetilde{J_s}}{\hat{a}^2(\hat{a}^2 + s_2^2)} && \text{(B.9)} \\
N' &= -\frac{\delta \widetilde{F_{zs}}}{\hat{a}^2(\hat{a}^2 + s_2^2)}, \qquad O' = 0, \qquad P' = 0 && \text{(B.10)} \\
Q' &= \frac{(i m_s g + s_2^2)\widetilde{J_s}}{s_2^2(\hat{a}^2 + s_2^2)}, \qquad R' = -\frac{\delta \widetilde{F_{zs}}}{s_2^2(\hat{a}^2 + s_2^2)}. && \text{(B.11)}
\end{aligned}
$$

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Figure Captions

Figure 1: Contour plot showing ω_1^2/ω_2^2 (solid lines) as a function of λ_H and λ_z for $N_B = 0.02$ s^{-1}, $\gamma = 1.4$, $g = 9.8$ m/s^2, $c_s = 310$ m/s and $\mathcal{H} = 7$ km. The right hand y axis shows the corresponding values of $m\mathcal{H}$. Dashed lines show c_H/c_s in intervals of 50 m/s.

Figure 2: a): Thin solid line shows u' for all GWs. Solid, dash, and dash-dot lines show $25w'/max(w')$ for the GWs with $\lambda_z = -10, -100$, and -300 km, respectively. Here, w' is calculated from Eq. (114), $max(w')$ is the maximum of $|w'|$, and $max(w')$ is 0.49, 2.5, and 2.2 m/s, respectively. The dotted line shows $25w'/max(w')$ for the Boussinesq solution given by Eq. (121), with $max(w') = 0.49$ m/s. b): Thin solid line shows u' and dash-dot line shows $25w'/max(w')$ for the GW with $\lambda_z = -300$ km. The dotted line shows a pure cosine curve, $26\cos(\Phi - kx)$, with $\Phi = -55^o$.

Figure 3: Phase relationship between u', $T'/\overline{T}$, and $\rho'/\overline{\rho}$ for the same GWs from Fig. 2. a): Thin solid line shows u' for all GWs. Solid, dash, and dash-dot lines shows $800T'/\overline{T}$ for the GWs with $\lambda_z = -10$, -100, and -300 km, respectively. Here, $T'/\overline{T}$ is calculated from Eq. (114) and (117). The dotted line shows the Boussinesq solution given by Eqs. (121) and (124). b): Same as in a), but the solid, dash, and dash-dot lines shows $800\rho'/\overline{\rho}$ using Eqs. (114) and (116). The dotted line shows the Boussinesq solution given by Eqs. (121) and (123). c): $100T'/\overline{T}$ (%) (solid line) and $100\rho'/\overline{\rho}$ (%) (dashed line) for the GW with $\lambda_z = -10$ km. d): Same as c), but for the GW with $\lambda_z = -300$ km.

Table 1: Symbols and notation

$F_x(\mathbf{x}),\ F_y(\mathbf{x}),\ F_z(\mathbf{x})$	body force in x, y, and z directions
J	heating
$\mathcal{F}(t)$	temporal evolution of forcing/heating
χ	total forcing/heating duration
$\hat{a} = 2\pi n/\chi$	frequency of forcing/heating
n	number of cycles in forcing/heating
$p = \mathrm{r}\rho T$	ideal gas law
$(\xi, \sigma, \eta) = \mathrm{e}^{-z/2\mathcal{H}}(u', v', w')$	scaled velocity perturbation
$\phi = \mathrm{e}^{-z/2\mathcal{H}}\rho'/\overline{\rho}$	scaled density perturbation
$\psi = \mathrm{e}^{-z/2\mathcal{H}}p'/\overline{\rho}$	scaled pressure perturbation
$\zeta = \mathrm{e}^{-z/2\mathcal{H}}T'/\overline{T}_0$	scaled temperature perturbation
$(F_{xs}, F_{ys}, F_{zs}) = \mathrm{e}^{-z/2\mathcal{H}}(F_x, F_y, F_z)$	scaled body force
$J_s = \mathrm{e}^{-z/2\mathcal{H}}\ \mathrm{r}J$	scaled heating
$c_s = \sqrt{\gamma g \mathcal{H}} = \sqrt{\gamma \mathrm{r} \overline{T}_0}$	sound speed
$N_B = \sqrt{\gamma - 1} g/c_s$	buoyancy frequency
(k, l, m)	wavenumber vector
$\omega_{Ir},\ \omega_r$	intrinsic and observed wave frequencies
$s = i\omega_{Ir},\ s_r = i\omega_r$	Laplace transform variables
$\omega_1 = \omega_{GW}$	intrinsic GW frequency
$\omega_2 = \omega_{AW}$	intrinsic AW frequency
subscripts "IV", "FH"	Initial value and Forced/heated solutions
$\mathcal{S} = \sin\omega t + \sin\omega(\sigma - t)$	$\mathcal{C} = \cos\omega t - \cos\omega(\sigma - t)$

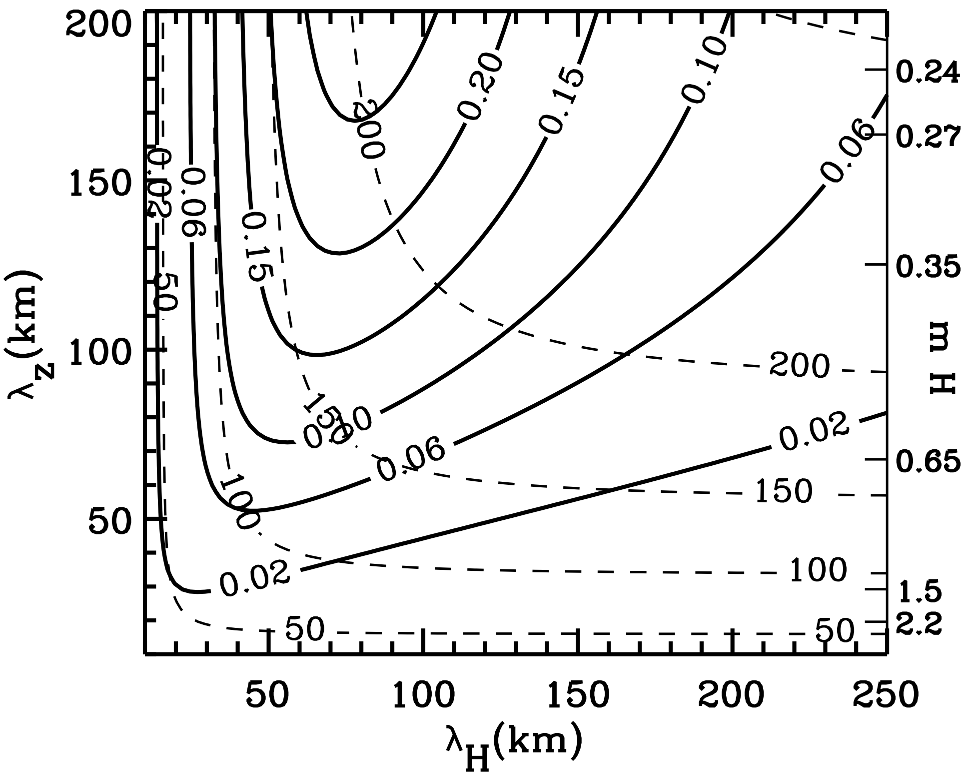

Figure 1:

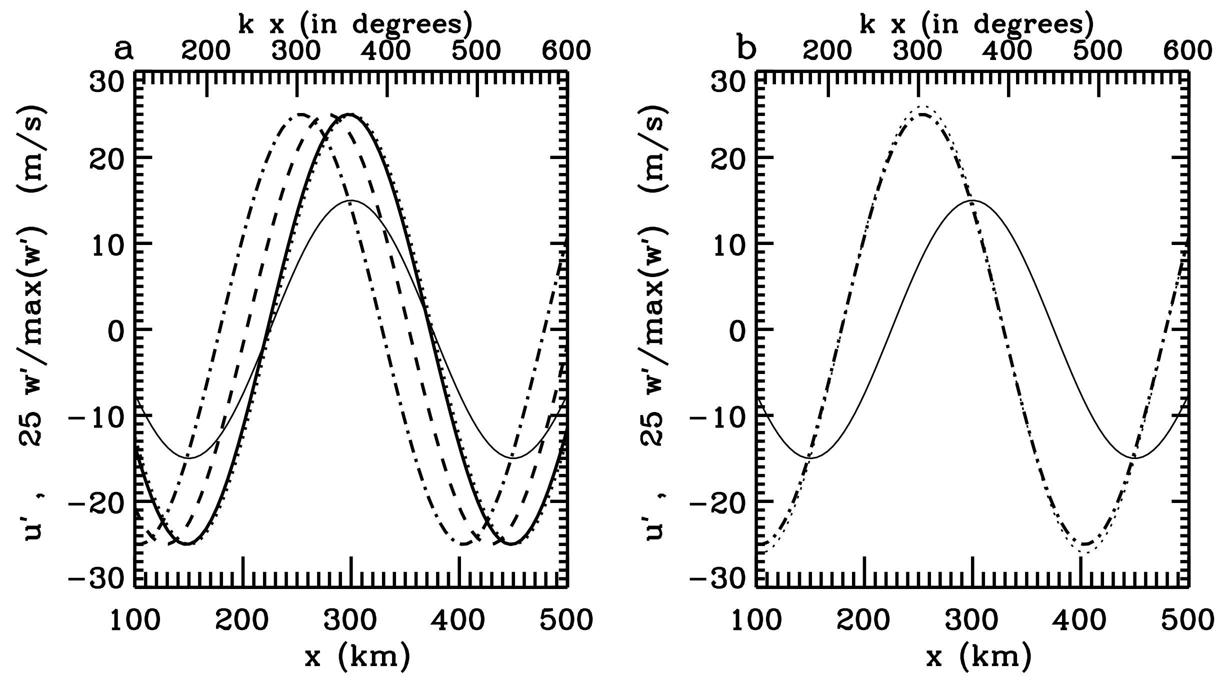

Figure 2:

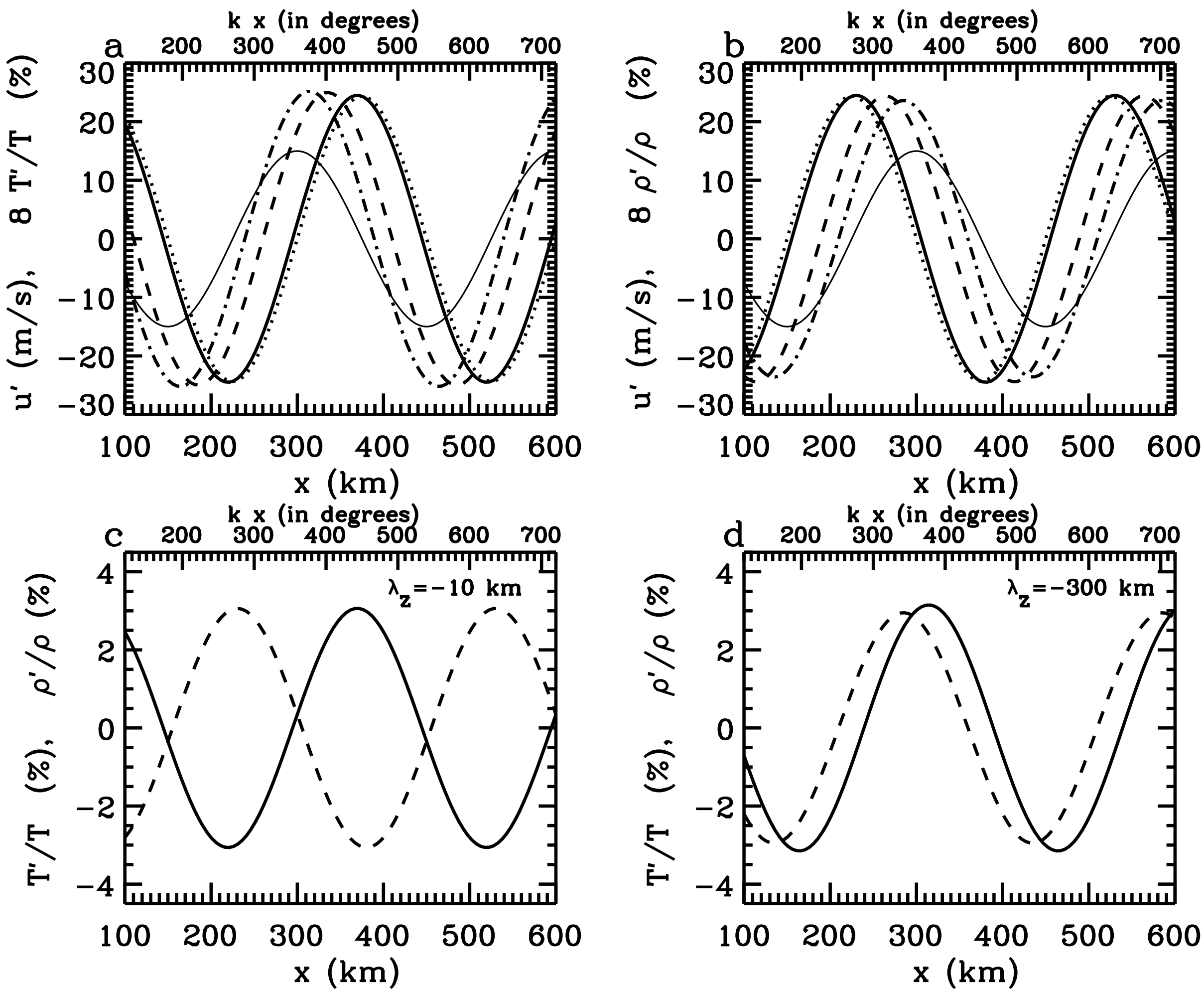
a
b
c
d
k x (in degrees)
200 300 400 500 600 700
u' (m/s), 8 T'/T (%)
u' (m/s), 8 ρ'/ρ (%)
T'/T (%), ρ'/ρ (%)
λz=−10 km
λz=−300 km
x (km)
100 200 300 400 500 600

Figure 3: