ASEN 5151: High Speed Aerodynamics

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HW#2 Solution

1. The thrust developed by a nozzle is given by

$$T = \rho_e u_e^2 A_e + (p_e - p_{atm}) A_e, \tag{1}$$

where the subscript e denotes conditions at the nozzle exit and where p_{atm} is the local atmospheric pressure. The thrust equation is put in non-dimensional form by dividing by $p_0 A_e$, where p_0 is the upstream stagnation pressure. The first term on the right hand is also multiplied and divided by the sound speed squared, $a_e^2 = \gamma R T_e$ to give

$$\frac{T}{p_0 A_e} = \left(\frac{u_e^2}{a_e^2}\right) \gamma \left(\frac{\rho_e R T_e}{p_0}\right) + \left(\frac{p_e}{p_0} - \frac{p_{atm}}{p_0}\right). \tag{2}$$

Noting the definition of the Mach number as well as the fact that $\rho_e R T_e = p_e$ by the ideal gas law, the above equation can be written as

$$\frac{T}{p_0 A_e} = \gamma M_e^2 \left(\frac{p_e}{p_0}\right) + \left(\frac{p_e}{p_0} - \frac{p_{atm}}{p_0}\right). \tag{3}$$

The nozzle exit pressure will equal the local atmospheric pressure unless the nozzle is either under- or over-expanded. When the pressure is matched there is no pressure contribution to the thrust and the non-dimensional thrust depends only on the exit to stagnation pressure ratio and the exit Mach number.

The area-Mach number relation (Eq. (8.43) in Bertin and Cummings) yields both a subsonic and a supersonic solution for $A_e/A_t = 8$. It is not possible to solve for the Mach number directly, but a table such as 8.1 in Bertin and Cummings, an online calculator such as `http://www.dept.aoe.vt.edu/~devenpor/aoe3114/calc.html`, or a small program using an iteration scheme can be used to find the values. In any case the values are $M = 0.072567$ and $M = 3.6772$. Given the Mach number, the corresponding pressure ratios are computed via Eq. (8.36) in Bertin and Cummings, giving $p_e/p_0 = 0.99632$ and $p_e/p_0 = 0.010221$. These values along with the resulting thrust are shown in Table 2.

If the atmospheric pressure is slightly below $0.99632 p_0$ the flow will accelerate to supersonic speeds downstream of the throat. A shock will then form in the nozzle at the position required to produce an exit pressure that matches the local atmospheric pressure. The easiest way to generate thrust data for this flow regime is to pick a position for the shock and then compute the resulting exit pressure (which will also equal the atmospheric pressure). The steps are:

- Assume a shock position in the form $1 \le A_s/A_t \le 8$.
- Compute the Mach number and pressure on the upstream side of the shock via Eqs. (8.43) and (8.36). Call these M_1 and p_1/p_0.

- Compute the Mach number and pressure on the downstream side of the shock via Eqs. (8.75) and (8.72) (with $\theta = 0$). Call these M_2 and p_2/p_1.

- Compute the exit Mach number by repeated application of Eq. (8.43). This is possible since the flow is isentropic from the downstream side of the shock to the nozzle exit. Writing the inverse of Eq. (8.43) symbolically as $A/A^* = f(M)$, we can evaluate this relation for station 2 (downstream side of the shock) and station e (the nozzle exit) and then ratio the two expressions to get

$$\frac{A_e}{A_2} = \frac{f(M_e)}{f(M_2)}. \tag{4}$$

Since $A_2 = A_s$, the Area ratio A_e/A_2 is

$$\frac{A_e}{A_2} = \frac{A_e}{A_s} = \left(\frac{A_e}{A_t}\right)\left(\frac{A_t}{A_s}\right) = \frac{8}{(A_s/A_t)}. \tag{5}$$

Thus the exit Mach number is determined from

$$f(M_e) = \frac{8}{(A_s/A_t)} f(M_2). \tag{6}$$

- Compute the exit pressure via a similar repeated application of Eq. (8.36). Let $p/p_0 = g(M)$ then

$$p_e/p_2 = g(M_e)/g(M_2). \tag{7}$$

- Compute the exit pressure normalized by the upstream stagnation pressure via

$$\frac{p_e}{p_0} = \left(\frac{p_1}{p_0}\right)\left(\frac{p_2}{p_1}\right)\left(\frac{p_e}{p_2}\right). \tag{8}$$

As an example of this procedure, let $A_s/A_t = 2$. For this area ratio Eq. (8.43) gives $M_1 = 2.1972$. Using this Mach number, Eq. (8.36) gives $p_1/p_0 = 0.093933$. Given M_1, Eq. (8.75) determines the Mach number on the downstream side of the shock as $M_2 = 0.54743$, and Eq. (8.72) gives the pressure ratio as $p_2/p_1 = 5.4656$. Given M_2, Eq. (6) above, along with Eq. (8.43) yields $M_e = 0.11586$. Equation (7) above, along with (8.72) then gives the exit pressure as $p_e/p_2 = 1.2145$. The required exit to stagnation pressure ratio is then found from Eq. (8) as $p_e/p_0 = 0.62353$. Finally, Eq. (3) gives the non-dimensional thrust as $T/(p_0 A_e) = 0.011718$. These values, along with sets for other shock positions are shown in Table 1.

Once the atmospheric pressure drops below $0.15954 p_0$ (the exit pressure when the shock sits at the exit), any shock or expansion fan required to adjust the exit pressure to the local atmospheric pressure must occur outside the nozzle. The flow within the nozzle (including the exit plane) is thus fixed at the supersonic isentropic solution, independent of the atmospheric pressure. The momentum component of the thrust is also fixed and any remaining change in thrust due to the pressure component. A few values for thrust for this regime are shown in Table 2.

The thrust distribution is plotted as a function of the driving pressure ratio (p_0/p_{atm}) in Figure 1.

A_s/A_t	A_e/A_s	M_1	p_1/p_0	M_2	p_2/p_1	M_e	p_e/p_2	p_e/p_0	$T/(p_0A_e)$
2.0	4.0000	2.1972	0.093933	0.54743	5.4656	0.11586	1.2145	0.62353	0.011718
3.0	2.6667	2.6374	0.047299	0.50069	7.9486	0.16479	1.1645	0.43780	0.016643
4.0	2.0000	2.9402	0.029787	0.47883	9.9188	0.21515	1.1328	0.33468	0.021689
5.0	1.6000	3.1748	0.020993	0.46561	11.592	0.26736	1.1039	0.26865	0.026885
6.0	1.3333	3.3679	0.015841	0.45657	13.066	0.32220	1.0736	0.22222	0.032297
7.0	1.1429	3.5328	0.012515	0.44990	14.394	0.38077	1.0396	0.18728	0.038014
8.0	1.0000	3.6772	0.010221	0.44471	15.609	0.44471	1.0000	0.15954	0.044172

Table 1: Solutions for shocks within the nozzle at the indicated position A_s/A_t.

condition	p_{atm}/p_0	p_e/p_0	M_e	$T/(p_0A_e)$
no flow	1.0000	1.0000	0.0000	0.0000
sonic throat, isentropic subsonic exit	0.99632	0.99632	0.072567	0.0073452
shock at $A_s/A_e = 2$	0.62353	0.62353	0.11586	0.011718
shock at $A_s/A_e = 3$	0.43780	0.43780	0.16479	0.016643
shock at $A_s/A_e = 4$	0.33468	0.33468	0.21515	0.021689
shock at $A_s/A_e = 5$	0.26865	0.26865	0.26736	0.026885
shock at $A_s/A_e = 6$	0.22222	0.22222	0.32220	0.032297
shock at $A_s/A_e = 7$	0.18728	0.18728	0.38077	0.038014
shock at $A_s/A_e = 8$	0.15954	0.15954	0.44471	0.044172
overexpanded	0.15954	0.010221	3.6772	0.044172
overexpanded	0.12221	0.010221	3.6772	0.081502
overexpanded	0.084879	0.010221	3.6772	0.11883
overexpanded	0.047550	0.010221	3.6772	0.15616
overexpanded	0.010221	0.010221	3.6772	0.19349
isentropic supersonic exit matched to atmospheric pressure	0.010221	0.010221	3.6772	0.19349
underexpanded	0.0076656	0.010221	3.6772	0.19604
underexpanded	0.0051104	0.010221	3.6772	0.19860
underexpanded	0.0025552	0.010221	3.6772	0.20115
underexpanded	0.0000	0.010221	3.6772	0.20371

Table 2: Nozzle flow solutions for various flow regimes.

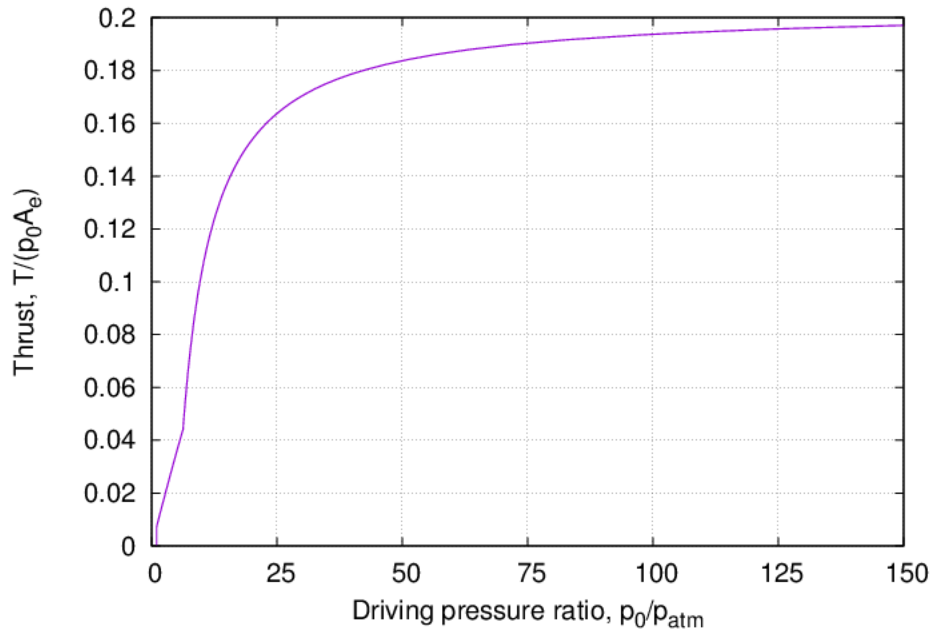


Figure 1: Thrust for a nozzle with $A_e/A_t = 8$ as a function of the driving pressure ratio.

2. The oblique shock relation is given by Eq. (4.17) in Anderson

$$\tan\theta = 2\cot\beta\left[\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2}\right], \tag{9}$$

where M_1 is the upstream Mach number, β is the shock angle and θ is the flow deflection angle. This equation shows that as $\beta \to \mu = \sin^{-1}(1/M_1)$, $\theta \to 0$. Thus for small θ it is appropriate to take

$$\beta = \mu + \epsilon, \tag{10}$$

where ϵ is a small angle. The trigonometric terms above are approximated as follows

$$\begin{aligned}
\tan\theta &\simeq \theta, & (11)\\
\sin\beta &= \sin(\mu+\epsilon) = \sin\mu\cos\epsilon + \cos\mu\sin\epsilon \simeq \sin\mu + \epsilon\cos\mu, & (12)\\
\sin^2\beta &\simeq (\sin\mu + \epsilon\cos\mu)^2 \simeq \sin^2\mu + 2\epsilon\sin\mu\cos\mu, & (13)\\
\cos\beta &= \cos(\mu+\epsilon) = \cos\mu\cos\epsilon - \sin\mu\sin\epsilon \simeq \cos\mu - \epsilon\sin\mu, & (14)\\
\cos 2\beta &= \cos(2\mu + 2\epsilon) = \cos 2\mu\cos 2\epsilon - \sin 2\mu\sin 2\epsilon \simeq \cos 2\mu - 2\sin 2\mu\epsilon & (15)\\
&\simeq \cos^2\mu - \sin^2\mu - 4\epsilon\sin\mu\cos\mu, & (16)\\
\cot\beta &= \frac{\cos\beta}{\sin\beta} \simeq \frac{\cos\mu - \epsilon\sin\mu}{\sin\mu + \epsilon\cos\mu} = \frac{\cot\mu - \epsilon}{1 + \epsilon\cot\mu} \simeq (\cot\mu - \epsilon)(1 + \epsilon\cot\mu) & (17)\\
&\simeq \cot\mu + (1 + \cot^2\mu)\epsilon = \cot\mu + \epsilon\csc^2\mu. & (18)
\end{aligned}$$

From the Mach triangle in Figure 2 we have

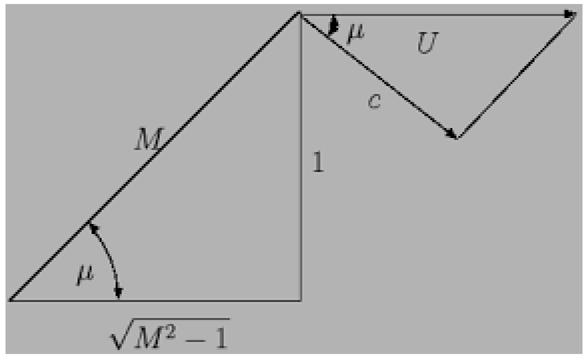


Figure 2: Mach triangle.

$$\sin\mu = \frac{1}{M_1}, \tag{19}$$

$$\cos\mu = \frac{\sqrt{M_1^2-1}}{M_1}, \tag{20}$$

$$\csc\mu = \frac{1}{\sin\mu} = M_1. \tag{21}$$

$$\tag{22}$$

These results are used to simplify the trigonometric relationships above, yielding

$$\sin\beta \simeq \frac{1}{M_1}\left(1+\epsilon\lambda\right), \tag{23}$$

$$\cos 2\beta \simeq \frac{1}{M_1^2}\left[(M_1^2-2)-4\epsilon\lambda\right], \tag{24}$$

$$\cot\beta \simeq \lambda-\epsilon M_1^2, \tag{25}$$

where

$$\lambda = \sqrt{M_1^2-1}. \tag{26}$$

Now returning to the oblique shock relation with the above approximations and neglecting any subsequently-generated terms of order ϵ^2 or higher allows the development

$$\theta \simeq \left(2\lambda - \epsilon M_1^2\right)\left[\frac{(1+2\lambda\epsilon)-1}{\gamma M_1^2 + M_1^2 - 4\epsilon\lambda + 2}\right] \tag{27}$$

$$\simeq \frac{4\lambda^2\epsilon}{(\gamma+1)M_1^2\left[1 - \frac{4\lambda\epsilon}{(\gamma+1)M_1^2}\right]} \tag{28}$$

$$\simeq \frac{4\lambda^2\epsilon}{(\gamma+1)M_1^2}\left[1 + \frac{4\lambda\epsilon}{(\gamma+1)M_1^2}\right] \tag{29}$$

$$\simeq \frac{4\lambda^2\epsilon}{(\gamma+1)M_1^2} \tag{30}$$

$$\simeq \left[\frac{4\left(M_1^2-1\right)}{(\gamma+1)M_1^2}\right]\epsilon, \tag{31}$$

which implies

$$\beta = \mu + \epsilon = \mu + \left[\frac{(\gamma+1)M_1^2}{4\left(M_1^2-1\right)}\right]\theta. \tag{32}$$

Thus, to leading order, the shock angle is increased from the Mach angle by an amount proportional to the flow deflection angle. The Mach number normal to the shock is

$$M_{1n} = M_1 \sin\beta \tag{33}$$

$$\simeq M_1\left[\frac{1}{M_1}(1+\epsilon\lambda)\right] \tag{34}$$

$$\simeq 1 + \sqrt{M_1^2-1}\left[\frac{(\gamma+1)M_1^2}{4\left(M_1^2-1\right)}\right]\theta \tag{35}$$

$$\simeq 1 + \left[\frac{(\gamma+1)M_1^2}{4\sqrt{M_1^2-1}}\right]\theta. \tag{36}$$

Again, to leading order, this result shows that the component of Mach number normal to the shock is greater than 1 by an amount proportional to the flow deflection angle.

Both the results for the shock angle and the normal component of Mach number have terms proportional to $M_1^2 - 1$ in the denominator. Thus as $M_1 \to 1$ the approximations diverge. Physically this mathematical consequence stems from the fact that maximum flow deflection angle for a $M_1 = 1$ shock is zero, while the wave angle is 90^0. A more careful analysis would show that the maximum flow deflection angle approaches zero faster than $M_1^2 - 1$. This behavior resolves the singularities in both Eqs. (32) and (36), and gives the physically correct results that $\beta = \mu$ and $M_{1n} = 1$ when $M_1 = 1$.

3. The change in entropy across a shock wave is given by the equation above (3.60) in Anderson

$$\Delta s = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right). \tag{37}$$

Using the ideal gas law in the form $T_2/T_1 = (p_2/p_1)/(\rho_2/\rho_1)$ as well as the relations $R = c_p - c_v$, $\gamma = c_p/c_v$, the entropy change can be written equivalently as

$$\frac{\Delta s}{c_v} = \ln\left(\frac{p_2}{p_1}\right) - \gamma\ln\left(\frac{\rho_2}{\rho_1}\right). \tag{38}$$

The density ratio across a shock wave is given by Eq. (3.53) in Anderson

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}. \tag{39}$$

In order to investigate the behavior for M_1 slightly greater than 1, set $M_1 = 1+\epsilon$. As a consequence $M_1^2 = 1+2\epsilon+\epsilon^2$ and thus the expression for the density ratio can be written as

$$\begin{aligned}
\frac{\rho_2}{\rho_1} &= \frac{(\gamma+1)(1+2\epsilon+\epsilon^2)}{2+(\gamma-1)(1+2\epsilon+\epsilon^2)} && (40)\\
&= \frac{(\gamma+1)(1+2\epsilon+\epsilon^2)}{(\gamma+1)\left[1+2\left(\frac{\gamma-1}{\gamma+1}\right)\epsilon+\left(\frac{\gamma-1}{\gamma+1}\right)\epsilon^2\right]} && (41)\\
&= \frac{(1+2\epsilon+\epsilon^2)}{(1+2A\epsilon+A\epsilon^2)}, && (42)
\end{aligned}$$

where

$$A = \frac{\gamma-1}{\gamma+1}. \tag{43}$$

Making use of the binomial expansion

$$\frac{1}{1+\delta} = 1 - \delta + \delta^2 - \delta^3 + \ldots, \tag{44}$$

the expression for the density ratio can be rewritten as

$$\begin{aligned}
\frac{\rho_2}{\rho_1} &\simeq \left(1+2\epsilon+\epsilon^2\right)\left[1-\left(2A\epsilon+A\epsilon^2\right)+\left(2A\epsilon+A\epsilon^2\right)^2-\left(2A\epsilon+A\epsilon^2\right)^3\right] && (45)\\
&\simeq 1+2(1-A)\epsilon+(1-A)(1-4A)\epsilon^2-4A(1-A)(1-2A)\epsilon^3. && (46)
\end{aligned}$$

The expression for the entropy change requires the logarithm of the density ratio. Before doing this step it is useful to derive a general result that can be used for the logarithm of the pressure ratio as well. Starting with the Taylor series expansion of the logarithm about unity, we have

$$\ln(1+\delta) = \delta - \frac{1}{2}\delta^2 + \frac{1}{3}\delta^3 + \ldots. \tag{47}$$

Noting that the expression for the density ratio is of a form equivalent to the argument of the logarithm above, in which the small quantity δ can written symbolically as

$$\delta = c_1\epsilon + c_2\epsilon^2 + c_3\epsilon^3, \tag{48}$$

we can write

$$\ln(1+\delta) = (c_1\epsilon + c_2\epsilon^2 + c_3\epsilon^3) - \frac{1}{2}(c_1\epsilon + c_2\epsilon^2 + c_3\epsilon^3)^2 + \frac{1}{3}(c_1\epsilon + c_2\epsilon^2 + c_3\epsilon^3)^3 + \ldots . \tag{49}$$

After performing the products and retaining terms only out to ϵ^3 we have

$$\ln(1 + c_1\epsilon + c_2\epsilon^2 + c_3\epsilon^3) \simeq c_1\epsilon + \left(c_2 - \frac{1}{2}c_1^2\right)\epsilon^2 + \left(c_3 - c_1c_2 + \frac{1}{3}c_1^3\right)\epsilon^3. \tag{50}$$

Application of this rule to Eq. (46) with $c_1 = 2(1-A)$, $c_2 = (1-A)(1-4A)$, $c_3 = -4A(1-A)(1-2A)$ yields

$$\ln\left(\frac{\rho_2}{\rho_1}\right) \simeq (1-A)\left[2\epsilon - (1+2A)\epsilon^2 + \frac{2}{3}(1+A+4A^2)\epsilon^3\right]. \tag{51}$$

The pressure ratio across a shock is given by Eq. (3.57) in Anderson

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right). \tag{52}$$

Making use of the replacement $M_1 = 1+\epsilon \implies M_1^2 - 1 = 2\epsilon + \epsilon^2$, the above equation can be written as

$$\frac{p_2}{p_1} = 1 + 2B\epsilon + B\epsilon^2, \tag{53}$$

where

$$B = \frac{2\gamma}{\gamma+1}. \tag{54}$$

Note that the pressure ratio expression is still exact as no approximations have been made up to this point. We can now apply Eq. (50) with $c_1 = 2B$, $c_2 = B$, $c_3 = 0$ to get

$$\ln\left(\frac{p_2}{p_1}\right) \simeq B\left[2\epsilon - (2B-1)\epsilon^2 + \frac{2}{3}(4B-3)\epsilon^3\right]. \tag{55}$$

By comparing Eqs. (43) and (54) it is seen that A and B are related via

$$B = \gamma(1-A). \tag{56}$$

This result allows to write Eq. (55) as

$$\ln\left(\frac{p_2}{p_1}\right) \simeq \gamma(1-A)\left[2\epsilon - (1+2A)\epsilon^2 - \frac{2}{3}\gamma(1-A)(3-4\gamma(1-A))\epsilon^3\right]. \tag{57}$$

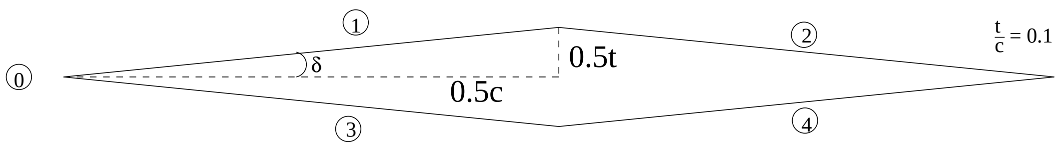
Now returning to Eq. (38) with the results of Eqs. (51) and (57) we have

$$\begin{aligned}\frac{\Delta s}{c_v} &\simeq \gamma(1-A)\left[2\epsilon-(1+2A)\epsilon^2-\frac{2}{3}\gamma(1-A)[3-4\gamma(1-A)]\epsilon^3\right]- \\ &\quad \gamma(1-A)\left[2\epsilon-(1+2A)\epsilon^2+\frac{2}{3}(1+A+4A^2)\epsilon^3\right]. \qquad (58)\end{aligned}$$

The ϵ and ϵ^2 terms are seen to cancel completely, leaving

$$\frac{\Delta s}{c_v} \simeq \frac{8}{3}\gamma AB\epsilon^3 = \frac{16}{3}\left(\frac{\gamma}{\gamma+1}\right)A\epsilon^3 = \frac{16}{3}\left(\frac{\gamma(\gamma-1)}{(\gamma+1)^2}\right)\epsilon^3. \qquad (59)$$

This is a very important result. It indicates that the entropy rises only as the third power of the fraction of Mach number in excess of 1. Thus shock losses are quite small for weak shocks (where M_{1n} is only slightly greater than 1). Conversely, the losses become enormous for strong shocks (where $M_{1n} > 2$ say).



4. The symmetric diamond airfoil is shown above. The wedge angle δ is computed via

$$\delta = \tan^{-1}\left(\frac{0.5t}{0.5c}\right) = \tan^{-1}\left(\frac{t}{c}\right) = \tan^{-1}(0.1) = 5.7106^0. \qquad (60)$$

Prandtl-Meyer (P-M) theory requires the change in the flow angle for each segment of the upper and lower surfaces. Here we use the linear flow analysis convention that a compression surface has a positive flow deflection angle and an expansion surface has a negative angle. Unfortunately this convention is opposite to that used for P-M analysis where an a positive deflection angle is used for an expansion surface. We account for the discrepancy by listing $-\Delta\theta$ in the table below and then use this quantity as a direct input to the P-M analysis. Under our convention, the flow angle is related to the surface slope and the angle of attack separately on the upper and lower surfaces, viz

$$\theta_u = \tan^{-1}\left(\frac{dy_u}{dx}\right) - \alpha, \qquad (61)$$

$$\theta_l = -\tan^{-1}\left(\frac{dy_l}{dx}\right) + \alpha. \qquad (62)$$

Solution to the Prandtl-Meyer equation yields the Mach number for each airfoil segment. The pressure is then computed from the following isentropic relation, obtained by ratioing the results of Eq. (8.36), evaluated at (p, M) and (p_∞, M_∞)

$$\frac{p}{p_\infty} = \left[\frac{1+0.5(\gamma-1)M_\infty^2}{1+0.5(\gamma-1)M^2}\right]^{\left(\frac{\gamma}{(\gamma-1)}\right)}. \qquad (63)$$

station	dy/dx	$\tan^{-1}(dy/dx)$ (deg)	-$\Delta\theta$ (deg)	ν (deg)	M	p/p_∞
0	0.0000	0.0000	0.0000	26.380	2.0000	1.00000
1	0.1000	5.7106	-3.7106	22.669	1.8677	1.22735
2	-0.1000	-5.7106	11.4212	34.090	2.2923	0.633292
0	0.0000	0.0000	0.0000	26.380	2.0000	1.00000
3	-0.1000	-5.7106	-7.7106	18.669	1.7293	1.51646
4	0.1000	5.7106	11.4212	30.090	2.1373	0.807151

Table 3: Solution to the symmetric diamond airfoil using Prandtl-Meyer theory. $t/c = 0.1$, $M_\infty = 2.0$, $\alpha = 2^0$.

Starting with a free-stream Mach number $M_\infty = 2.0$ and angle of attack of $\alpha = 2^0$ we can fill out Table 3. Here we have listed the flow deflection angle as $\Delta\theta$ to emphasize the fact that it is the change in the flow angle from segment to segment. The net pressure force normal to the airfoil axis is

$$n = \int_0^c (p_l - p_u)\,dx, \tag{64}$$

or in non-dimensional form

$$\frac{n}{1/2\rho_\infty u_\infty^2 c} = c_n = \frac{p_\infty}{1/2\rho_\infty u_\infty^2}\int_0^1 \left(\frac{p_l}{p_\infty} - \frac{p_u}{p_\infty}\right) d\frac{x}{c} \;=\; \frac{2}{\gamma M_\infty^2}\int_0^1 \left(\frac{p_l}{p_\infty} - \frac{p_u}{p_\infty}\right) d\left(\frac{x}{c}\right) \tag{65}$$

. The contributions to the integral are constant over each section, while the surface element $\Delta x/c = 0.5$ for all segments. Thus the normal force coefficient can be evaluated to give

$$\begin{aligned} c_n &= \frac{2}{\gamma M_\infty^2}\left[\frac{p_3}{p_\infty} + \frac{p_4}{p_\infty} - \frac{p_1}{p_\infty} - \frac{p_2}{p_\infty}\right]\frac{\Delta x}{c} && (66)\\ &= \frac{2}{1.4 * 2^2}\,[\,1.51646 + 0.807151 - 1.22735 - 0.633292\,]\,0.5 && (67)\\ &= 0.082673. && (68)\end{aligned}$$

A similar development leads to the axial force coefficient

$$\begin{aligned} c_a &= \frac{2}{\gamma M_\infty^2}\int_0^1 \left[\left(\frac{p_u}{p_\infty}\right)\left(\frac{dy_u}{dx}\right) - \left(\frac{p_l}{p_\infty}\right)\left(\frac{dy_l}{dx}\right)\right] d\left(\frac{x}{c}\right)\\ &= \frac{2}{\gamma M_\infty^2}\left[\left(\frac{p_1}{p_\infty}\right)\left(\frac{dy_1}{dx}\right) + \left(\frac{p_2}{p_\infty}\right)\left(\frac{dy_2}{dx}\right) - \left(\frac{p_3}{p_\infty}\right)\left(\frac{dy_3}{dx}\right) - \left(\frac{p_4}{p_\infty}\right)\left(\frac{dy_4}{dx}\right)\right]\frac{\Delta x}{c}\\ &= \frac{2}{1.4 * 2^2}\,[\,1.22735(0.1) + 0.633292(-0.1) - 1.51646(-0.1) - 0.807151(0.1)\,]\,0.5\\ &= 0.023274. && (69)\end{aligned}$$

The lift and drag coefficients are then computed from

$$c_l = c_n\cos\alpha - c_a\sin\alpha = 0.082673\cos(2^0) - 0.023274\sin(2^0) = 0.081810.$$

$$c_l = c_a\cos\alpha + c_n\sin\alpha = 0.023274\cos(2^0) + 0.082673\sin(2^0) = 0.026145.$$

station	dy/dx	$\tan^{-1}(dy/dx)$ (deg)	$\Delta\theta$ (deg)	β (deg)	ν (deg)	M	p/p_{left}	p/p_∞
0	0.0000	0.0000	0.0000	N.A.	26.380	2.0000	1.00000	1.000000
1	0.1000	5.7106	3.7106	33.132	22.652	1.8671	1.22741	1.22741
2	-0.1000	-5.7106	-11.4212	N.A.	34.073	2.2917	0.516056	0.633412
0	0.0000	0.0000	0.0000	N.A.	26.380	2.0000	1.00000	1.00000
3	-0.1000	-5.7106	7.7106	36.917	18.520	1.7242	1.51704	1.51704
4	0.1000	5.7106	-11.4212	N.A.	29.941	2.1317	0.532838	0.808337

Table 4: Solution to the symmetric diamond airfoil using shock-expansion theory. $t/c = 0.1$, $M_\infty = 2.0$, $\alpha = 2^0$.

We now repeat the problem using oblique shock relations for the compression surfaces. The pressure ratio across the shocks is computed from Eq. (8.72) in Bertin and Cummings. Equation (63) can no longer be used for the expansions since the shocks add a non-isentropic element. However, a similar equation can be used to compute the pressure ratio across the expansion itself, viz

$$\frac{p}{p_{left}} = \left[\frac{1+0.5(\gamma-1)M_{left}^2}{1+0.5(\gamma-1)M^2}\right]^{\left(\frac{\gamma}{(\gamma-1)}\right)}, \tag{70}$$

where the subscript $left$ refers to the conditions to the left (upstream) of the expansion corner. With this change, we must use a slightly more complicated relation to arrive at the required pressure ratio p/p_∞. For example, in order to compute the pressure on segment 2 of the airfoil, we proceed as follows

$$\frac{p_2}{p_\infty} = \left(\frac{p_1}{p_\infty}\right)\left(\frac{p_2}{p_1}\right). \tag{71}$$

The revised computation is shown in Table 4. Here we list the change in the flow angle ($\Delta\theta$) as being positive for a compression and negative for an expansion. The normal and axial force coefficients are then computed as

$$\begin{aligned}
c_n &= \frac{2}{\gamma M_\infty^2}\left[\frac{p_3}{p_\infty}+\frac{p_4}{p_\infty}-\frac{p_1}{p_\infty}-\frac{p_2}{p_\infty}\right]\frac{\Delta x}{c} && (72)\\
&= \frac{2}{1.4*2^2}\,[\,1.51704+0.808337-1.22741-0.633412\,]\,0.5 && (73)\\
&= 0.082956. && (74)\\
c_a &= \frac{2}{\gamma M_\infty^2}\left[\left(\frac{p_1}{p_\infty}\right)\left(\frac{dy_1}{dx}\right)+\left(\frac{p_2}{p_\infty}\right)\left(\frac{dy_2}{dx}\right)-\left(\frac{p_3}{p_\infty}\right)\left(\frac{dy_3}{dx}\right)+\left(\frac{p_4}{p_\infty}\right)\left(\frac{dy_4}{dx}\right)\right]\frac{\Delta x}{c}\\
&= \frac{2}{1.4*2^2}\,[\,1.22741(0.1)+0.633412(-0.1)-1.51704(-0.1)-0.808337(0.1)\,]\,0.5\\
&= 0.023263. && (75)
\end{aligned}$$

The lift and drag coefficients are then computed from

$$c_l = c_n \cos\alpha - c_a \sin\alpha = 0.082956 \cos(2^0) - 0.023263 \sin(2^0) = 0.082094.$$

$$c_l = c_a \cos\alpha + c_n \sin\alpha = 0.023263 \cos(2^0) + 0.082956 \sin(2^0) = 0.026143.$$

The estimated obtained from P-M theory compare very well to these exact values. The lift coefficient is in error by only 0.35% and the drag coefficient is almost perfect with a 0.0077% error.