MCEN 5021: Introduction to Fluid Dynamics

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Introduction to Tensor Notation

Tensor notation provides a convenient and unified system for describing physical quantities. Scalars, vectors, second rank tensors (sometimes referred to loosely as tensors), and higher rank tensors can all be represented in tensor notation. In the most general representation, a tensor is denoted by a symbol followed by a collection of subscripts, e.g.

$$X_j,\ \sigma_{ij},\ U_l,\ \beta_{klm},\ a,\ b,\ \alpha_{ijkl}$$

The number of subscripts attached to a tensor defines the *rank* of the tensor. (Note that the number of subscripts ranges from zero to four in the above examples.) A tensor of rank zero (no subscripts) is nothing more than a familiar scalar. A tensor of rank one (one subscript) is simply a common vector. If we are working a problem in three-dimensional space, then the vector will contain three components. In the tensor notation, these three components are represented by stepping the subscripted index through the values 1,2, and 3. As an example, suppose we are given the velocity vector in its common vector notation

$$\vec{U} = u\hat{e_x} + v\hat{e_y} + w\hat{e_z}$$

We may write this vector as a tensor of rank one as follows:

$$U_j \qquad (j = 1, 2, 3) \qquad \text{where} \quad U_1 = u \quad U_2 = v \quad U_3 = w$$

In most instances it is assumed that the problem takes place in three dimensions and clause $(j = 1, 2, 3)$ indicating the range of the index is omitted.

A tensor of rank two contains two free indices and thus bears some resemblance to a matrix. The similarity between a tensor of rank two and a matrix should be limited to a visual one, however, as each of the two objects have there own distinct mathematical properties. A tensor of rank two is sometimes written in vector notation as a symbol with two arrows above it. As we shall see, this usage should be limited to symmetric tensors. A symmetric tensor is invariant under an interchange of indices. That is $\sigma_{ij} = \sigma_{ji}$ for a symmetric tensor. As an example take the surface stress tensor. Since the surface stress is symmetric we may write the equivalence

$$\vec{\vec{\sigma}} \equiv \sigma_{ij} \qquad (i = 1, 2, 3 \quad j = 1, 2, 3)$$

A tensor of rank two with a range of three on both subscripts contains nine elements. These nine elements are formed by all possible permutations of the free indices. This is accomplished systematically by cycling the subscript i through 1, 2, and 3. At each value of the subscript i, the subscript j is cycled through 1,2,and 3. As an illustration, again consider the surface stress tensor

$$\overrightarrow{\overleftarrow{\sigma}} \equiv \sigma_{ij} = \left\{ \begin{matrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{matrix} \right\}$$

The top row is obtained by holding i fixed at 1 and cycling j through 1, 2, and 3. Similarly the second row is obtained by holding i fixed at 2 and then cycling through j. The procedure for the third row should now be obvious. In this case symmetry implies that $\sigma_{12} = \sigma_{21}$, $\sigma_{13} = \sigma_{31}$, and $\sigma_{23} = \sigma_{32}$.

Operations on Tensors

Contraction

The first fundamental operation on tensors is the contraction. Consider the common definition of a sum

$$\sum_{i=1}^{3} A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3$$

If we take A_i and B_i to be tensors of rank one (i.e. vectors), then the above operation defines a contraction over the free index i. Following a convention introduced by Einstein, the summation symbol is usually omitted. Under this so-called "Einstein summation convention", whenever a repeated index appears a summation over that index is implied. Thus the above example may be written as

$$A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3$$

Note that the above operation is equivalent to the dot product of the vectors $\vec{A}$ and $\vec{B}$.

$$\vec{A} \cdot \vec{B} \equiv A_k B_k = A_1 B_1 + A_2 B_2 + A_3 B_3$$

The choice for the name of the index to be summed over is arbitrary. For this reason it is sometimes referred to as a "dummy" index.

The contraction is defined between two (or more) tensors of unequal rank. For example consider the following example

$$w_i = u_j \sigma_{ij}$$

where

$$\begin{aligned} w_1 &= u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13} \\ w_2 &= u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23} \\ w_3 &= u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33} \end{aligned}$$

In general a contraction over the first index of a tensor of rank two will not equal a contraction over the second index. That is $u_i\sigma_{ij} \neq u_j\sigma_{ij}$. For the special case of a symmetric tensor $\sigma_{ij} = \sigma_{ji}$ and a contraction over either index yields the same result. That is $u_i\sigma_{ij} = u_j\sigma_{ij}$ provided σ_{ij} is symmetric. When this condition of symmetry exists, a vector notation may be used to represent the contraction as illustrated by the following example

$$(\vec{U} \cdot \overset{\Rightarrow}{\sigma}) \equiv U_j \sigma_{ij} \qquad (\text{provided} \quad \sigma_{ij} \quad \text{is symmetric})$$

It should be clear that the above definition relies on the symmetry of σ_{ij} since the dot product leaves an ambiguity over which of the two indices the contraction is to be made.

Divergence of a Tensor

The divergence of tensor is an application of index contraction. To see this, first define the spatial vector

$$\vec{x} \equiv x_i \quad \text{where} \quad x_1 = x \quad , \quad x_2 = y \quad , \text{and} \quad x_3 = z$$

The divergence of the velocity vector may then be represented as

$$\begin{aligned} \nabla \cdot \vec{U} \equiv \frac{\partial u_i}{\partial x_i} &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \\ &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \end{aligned}$$

The divergence of a higher rank tensor is likewise defined. If we restrict ourselves to symmetric tensors of rank two, the vector notation may also be used. Thus the divergence of the stress tensor can be written as

$$(\nabla \cdot \overset{\Rightarrow}{\sigma}) \equiv \tfrac{\partial \sigma_{ij}}{\partial x_j} = r_i$$

where

$$\begin{aligned}
r_1 &= \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} \\
r_2 &= \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} \\
r_3 &= \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3}
\end{aligned}$$

Note that the divergence operation lowers the rank of the tensor by one. Thus the divergence of a vector is a scalar and the divergence of a tensor of rank two is a tensor of rank one, which is a vector.

As another example of the contraction, consider the following work term from the energy equation

$$\begin{aligned}
\nabla \cdot (\vec{U} \cdot \vec{\vec{\sigma}}) \equiv \frac{\partial u_j \sigma_{ij}}{\partial x_i} &= \frac{\partial}{\partial x_1}(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13}) + \frac{\partial}{\partial x_2}(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) + \\
&\quad \frac{\partial}{\partial x_3}(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})
\end{aligned}$$

Gradient of a Tensor

Unlike the divergence operation, the gradient operation increases the rank of the tensor by one. Thus the gradient of a scalar is a vector, the gradient of a first rank tensor is a second rank tensor, and so on. The application of the gradient operator is straightforward. Study the following examples

$$\nabla \phi \equiv \frac{\partial \phi}{\partial x_i} = A_i$$

where

$$A_1 = \frac{\partial \phi}{\partial x_1} \quad A_2 = \frac{\partial \phi}{\partial x_2} \quad A_3 = \frac{\partial \phi}{\partial x_3}$$

$$\nabla \vec{U} \equiv \frac{\partial u_i}{\partial x_j} = \beta_{ij}$$

where

$$\beta_{ij} = \left\{ \begin{array}{ccc} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{array} \right\}$$

As a final example of both the contraction and the gradient operations, consider the convective term of the momentum equation

$$\vec{U} \cdot \nabla \vec{U} \equiv u_j \frac{\partial u_i}{\partial x_j} = s_i$$

where

$$\begin{aligned} s_1 &= u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} + u_3 \frac{\partial u_1}{\partial x_3} \\ s_2 &= u_1 \frac{\partial u_2}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} + u_3 \frac{\partial u_2}{\partial x_3} \\ s_3 &= u_1 \frac{\partial u_3}{\partial x_1} + u_2 \frac{\partial u_3}{\partial x_2} + u_3 \frac{\partial u_3}{\partial x_3} \end{aligned}$$

The Kronecker Delta

A second rank tensor of great utility is known as the Kronecker delta. It is defined as follows

$$\overset{\rightleftharpoons}{\delta} \equiv \delta_{ij} = \left\{ \begin{array}{ll} 1 & i = j \\ 0 & i \neq j \end{array} \right.$$

The Kronecker delta has some interesting and useful unitary properties. One such property is that the divergence of the Kronecker delta yields the gradient operator. That is

$$(\nabla \cdot \overset{\rightleftharpoons}{\delta}) \equiv \frac{\partial \delta_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j} \equiv \nabla$$